THE

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EDITED BY

T. A. A. BROADBENT, M.A.

WILLS HALL, BRISTOL, 9.

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(*An Association of Teachers and Students of Elementary Mathematics.*)

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EDITORIAL NOTE.

THE Editor, in performance of certain other duties, has been obliged to change his address ; communications should be addressed to him, *by name*, as follows :

T. A. A. Broadbent,
Wills Hall,
Bristol, 9.

It must not be assumed that this will necessarily remain the editorial address for the duration of the war, and readers will materially assist the production of the *Gazette* if they refer as occasion arises to the current number of the *Gazette*, which will contain the address available at the date of printing ; attention will of course be drawn to any change by means of a notice on the first page.

The dislocation due to this change is responsible for the delay in the appearance of this number. But it is hoped that the present arrangement of five numbers to a volume can be continued, and that the present volume will be completed in the normal fashion by a December number and a volume index ; the latter will probably be circulated with the *Gazette* for February 1941.

The Editor wishes to record here his obligation to Mr. F. W. Kellaway, whose assistance in the preparation of this number was generously offered and gratefully accepted.

THOMAS LITTLE HEATH

In the death of Sir Thomas Little Heath * on 16th March, at the age of seventy-eight, there passed away a great scholar whose literary achievements had been internationally recognised. During the first quarter of the present century he and Heiberg were the two leading authorities in the world in the field of Greek mathematics and astronomy. But Heiberg's profound erudition displayed, for example, in connection with his texts of Apollonius of Perga, Archimedes, Euclid, Ptolemy, Serenus, and Theodosius, put him in a class by himself.

When Heath's publications commenced to appear fifty-five years ago there was hardly anything in the English language about Greek mathematics. Gow's *A Short History of Greek Mathematics* had just been published and several chapters by G. J. Allman on *Greek Geometry from Thales to Euclid* had already appeared in *Hermathena* and these with later instalments were finally issued in book form in 1889. The first edition of the first volume of Cantor's monumental *Vorlesungen über Geschichte der Mathematik* (with over 340 pages on Greek mathematics) had been published in 1880, and summarised the work of earlier scholars. In this early period was also J. S. Mackay (1843-1914), a Scot, who for many years had given his holidays to the collation of manuscripts of the *Collection* of Pappus, and his nights to patient elucidation of the interpretation, until the work was finally complete, with a wealth of drawings. Just then (1876) the first volume of Hultsch's edition of Pappus came out and he discovered that all of his work, even to the smallest detail, had been anticipated. A little later he found himself forestalled in a less direct and complete way by the publication of Allman's work. His article on Euclid appeared in more than one edition of the *Encyclopædia Britannica*.

Before referring to outstanding features of Heath's achievements let us view a skeleton sketch of his career. He was born 5th October, 1861, at Barnetby-le-Wold, Lincolnshire, being the third son of Samuel Heath. One of Thomas's elder brothers, Robert Samuel Heath, was second wrangler in 1881 and afterwards professor of mathematics and vice-principal in the University of Birmingham. At school and at Trinity College, Cambridge, Thomas Heath read both classics and mathematics and took first class honours in both the classical tripos and the mathematical tripos, and he was twelfth wrangler in 1882. In 1884 he passed first in the open competition for the Home Civil Service, entered the Treasury and became

* Volume 2 of *Osiris*, 1936, was dedicated to Sir Thomas and contained an excellent portrait as frontispiece, an extended biographical sketch (pp. v-xxiii) by Professor D. E. Smith, and a bibliography of Heath's publications. A fine obituary notice by Sir D'Arcy W. Thompson appeared in *Nature*, 13th April, 1940, v. 145, pp. 578-579.

finally (1913-19) joint permanent secretary to the Treasury and auditor of the Civil List. In 1919 he was appointed comptroller of the National Debt Office, from which position he retired at the end of 1926, because of age limitations. Thus his regular duties kept him busy for seven hours a day for over forty years. In his interesting little book on *The Treasury* (1927) he described the many changes he had witnessed in his long career.

But Heath was occupied in other ways. He served as president of the Mathematical Association in 1922. Elected a fellow of the Royal Society in 1912, he was a member of its Council, 1920-21 and 1926-28. He was also one of the Cambridge Commissioners under the University of Oxford and Cambridge Act of 1923, and a member of the Royal Commission on National Museums and Galleries (1927-29).

Let us now turn to his extraordinary literary output, a product of his evening hours. These commenced to appear in print in 1885. Already we find " T. L. Heath, B.A.", shortly after graduation, contributing two articles to the *Encyclopædia Britannica*, an edition of which was then in process of publication. Since such articles are not listed in Professor Smith's bibliography,* it may be well as a matter of record to bring them together here. Ninth edition, vols. 18-19 (1885) ; eleventh edition (1910-11) ; fourteenth edition (1929). " Apollonius of Perga " (11th, 14th), " Archimedes " (11th, 14th), " Arithemius " (11th), " Diophantus " (11th, 14th), " Eratosthenes " (11th, 14th), " Euclid " (14th), " Heracleides Ponticus " (14th), " Heron of Alexandria " (11th, 14th), " Nicomachus " (14th), " Pappus " (9th, 11th, 14th), " Porism " (9th, 11th, 14th), " Ptolemy " (in part, 14th), " Pythagoras " (14th), " Serenus of Antissa " (11th, 14th), " Theodosius " (11th, 14th).

But, of other publications of 1885, apart from contributions to *Journal of Philology*, there was Heath's first book *Diophantus of Alexandria ; a Study in the History of Greek Algebra*. This was a dissertation, written with Glaisher's encouragement, which led to his election as Fellow of Trinity College. It was published by the Cambridge University Press on Cayley's recommendation. Two-thirds of the work were occupied with just the kind of material the scholarly student of Diophantus would like to have, and in the last third, an appendix, in an abstract of the Arithmetics and the tract on Polygonal Numbers. This youthful work, entirely recast by the mature scholar in its second edition just twenty-five years later, not only takes account of all pertinent new material which had become available, but presents also the work of Fermat and Euler which is intimately associated with that of Diophantus. The result was a volume of great importance.

Heath's second and third books came out in successive years 1896 and 1897 ; they were the Conic Sections of Apollonius of Perga,

* Another article is also omitted here, namely : " Geometry old and new ", *Nineteenth Century and After*, vol. 103, 1928, pp. 238-246.

" edited in modern notation, with introductions, including an essay on the earlier history of the subject " with notable help from Zeuthen's survey, and the Works of Archimedes, " edited in modern notation with introductory chapters ". The extraordinary discovery by Heiberg in an Istanbul library of the work of Archimedes on " The Method " led to Heath publishing a supplement to the latter work in 1912. (A German edition, with the supplement, appeared in 1914.) For the first time these treatises, taking account of the latest research, made it possible for the ordinary mathematician to gain a true appreciation of the wonderful achievements of these great Greeks. The condensation of texts from complete literal translations, into modern notation and abbreviated forms, while always retaining the spirit and lines of argument of the originals, was carried out in masterly fashion.

Eleven years later, 1908, a literal translation of Euclid's Elements, from the text of Heiberg, with introduction and commentary, was published in three volumes. This is a most remarkable work and constitutes a veritable encyclopedia of elementary geometry. It was a noble monument to be erected in a country where the Elements had been studied for a thousand years. That this large work was already out of print soon after the great war was over is of interest. A second edition, materially revised and somewhat enlarged, appeared in 1926. In 1920 Heath published another Euclid work of great merit, helpful alike to Greek and to Mathematics, *Euclid in Greek, Book I ; with Introduction and Notes.* " The notes are admirable. They contain, it is true, a good deal of elementary Greek for the weaker vessels ; but they also deal with the philosophy of the work and its foundations in a way which will be found valuable by many thoughtful men." (Review in *Classical Review*, v. 34, p. 180.)

Heath's next major work was *Aristarchus of Samos the ancient Copernicus, A History of Greek Astronomy to Aristarchus together with Aristarchus's Treatise on the Sizes and Distances of the Sun and Moon, a new Greek Text with Translation and Notes*, Oxford, 1913. It owed its inception to questions raised by an old school-fellow, Professor H. H. Turner of Oxford. Not only was one more remarkable geometrical work thus made available to the English student, but the validity of referring to Aristarchus as the Copernicus of Antiquity was well established. In 1932 this work was followed by another of a somewhat popular nature in the " Library of Greek Thought " series, his last published book, *Greek Astronomy*. After an Introduction of 44 pages there are 180 pages of translations, from different authors, of passages dealing with astronomical matters. These are somewhat in the style of the recently published volume of the Loeb Classical Library, *Selections illustrating the History of Greek Mathematics*, except that there the Greek text is given with the translations.

As soon as his Aristarchus volume was published Heath started the preparation of the splendid work which was to crown his achieve-

ments in connection with his "favourite hobby of Greek Mathematics", namely, the preparation of *A History of Greek Mathematics*. The bulk of this was "written as a distraction during the first three years of the war", and the manuscript was sent to the printer in October 1916; but the two volumes did not actually appear until 1921. This work was written both for the classical scholar and for the expert mathematician. To provide for the less expert intelligent reader who could dispense with considerable detail, Heath published in 1931 the single volume work, *A Manual of Greek Mathematics*, which contains corrections of slips in the larger work, changes due to advances in knowledge, and several pages on the remarkable discoveries of Neugebauer in connection with Babylonian mathematics.

All of Heath's writings are characterised by extraordinary clarity, especially noticeable in dealing with a complex mass of material, and remarkable accuracy, which could only be achieved by outstanding abilities coupled with exceptional lucidity of thought and relentless thoroughness in bibliographic surveys. His gift for literary expression, freed from all that was fanciful, was also most noticeable. That Heath's ten works should have been written and published simply as one of his hobbies, in an otherwise very busy life, seems almost unbelievable. Another occupation of his leisure was music (Bach's being a special comforter in times of spiritual trial), and yet another, mountaineering—he had made most of the principal ascents of the Dolomites.

When he was fifty-three Sir Thomas married Miss Ada Thomas, an accomplished pianist of professional standing, and they had a son and a daughter. He was created a Companion of the Bath in 1903, a Knight Commander of the Bath in 1909, and a Knight Commander of the Royal Victorian Order in 1916. He received the Sc.D. degree from Cambridge in 1896, that of Hon. Sc.D. from Oxford in 1913, and that of Hon. Litt.D. from Dublin in 1929. In 1920 he was made an honorary fellow of Trinity College, Cambridge. Everyone who had the privilege of meeting Sir Thomas must have been impressed by the delightful simplicity and courtesy of his bearing.

R. C. ARCHIBALD.

Brown University,
Providence, Rhode Island.

GLEANINGS FAR AND NEAR.

1321. We have no knowledge, that is, no general principles drawn from the contemplation of particular facts, but what has been built up by pleasure, and exists in us by pleasure alone. The Man of Science, the Chemist and Mathematician, whatever difficulties and disgusts they may have had to struggle with, know and feel this.—William Wordsworth, Preface to *Lyrical Ballads*, Vol. I. [Per Mr. A. F. Mackenzie.]

REMAINDERS IN INTERPOLATION AND QUADRATURE FORMULAE.

By P. J. Daniell.

The determination of the errors involved in interpolation or quadrature formulae often involves complicated reasoning. There are many cases, however, where the remainder can be obtained very easily. These cases belong to the class which we call simplex.

In general a formula will be expressible in the form

$$L(f) = \Sigma_r A_r f(x_r) + \Sigma_s B_s f'(x_s) + \Sigma_t C_t f''(x_t) + \dots \qquad \dots\dots(1)$$

to a finite number of terms and it is assumed throughout that $f(x)$ possesses derivatives as far as the remainder may require.

In Lagrange's formula

$$f(b) = \Sigma k_r f(x_r) + R,$$

that is,

$$L(f) = f(b) - \Sigma k_r f(x_r) = R;$$

in Simpson's rule, in terms of the integrated function,

$$L(f) = f(b) - f(a) - \tfrac{1}{6}(b-a)[f'(a) + 4f'(c) + f'(b)] = R,$$

where $c = \frac{1}{2}(a+b)$. In each case the remainder is $R = L(f)$.

A formula is said to be of order n if

$$L(x^q) = 0, \quad q = 0, 1, 2, \dots, n-1,$$

$$L(x^n) = E \neq 0.$$

For example the Lagrange formula with n points other than b is of order n and Simpson's rule is of order 5 in the integrated function.

A formula of order n is said to be *simplex* if, for every function $f(x)$ possessing derivatives up to the nth order in the enclosing interval,

$$L(f) = 0 \text{ implies } f^{(n)}(\xi) = 0$$

somewhere in the interval.

If a formula is simplex the remainder can be obtained directly.

Let $\lambda = L(f)/E, \quad g(x) = f(x) - \lambda x^n.$

Then $L(g) = 0$ and therefore, the formula being simplex,

$$g^{(n)}(\xi) = f^{(n)}(\xi) - \lambda n! = 0$$

somewhere in the interval. Thus

$$L(f) = E\lambda = \frac{E}{n!} f^{(n)}(\xi). \qquad \dots\dots\dots\dots\dots\dots(2)$$

This only requires the determination of $L(x^n)$, which is easy. In many cases we use, instead, $E = L(p_n)$ where p_n is some convenient polynomial whose leading term is x^n.

There remains the problem of deciding whether a formula is or is not simplex. This is always possible by expressing $L(f)$ as an

integral in terms of $f^{(n)}(x)$ and the formula is simplex if the coefficient of $f^{(n)}(x)$ in the integral does not change sign. But this is a complicated method and most cases yield to much simpler reasoning.

Direct inspection. In formula (1) let b be one of the points x_r and let the corresponding coefficient be $A \neq 0$. We can usually choose a polynomial p of degree $n-1$ so that the difference $g=f-p$ vanishes at each of the x_r other than b and so that $g'(x_s)=0$, $g''(x_t)=0, \dots$ also at the relevant points of the formula. In Simpson's rule it is also possible to make $g(c)=0$ in addition to making

$$g(a)=0=g'(a)=g'(b)=g'(c).$$

In Gauss' quadrature formulae it is similarly possible to make $g(x_s)=0$ as well as $g'(x_s)=0$, because if the number of points x_s for which $g'(x_s)$ appears in the formula is r the order of $L(f)$ is $2r+1$.

Now $$L(g)=L(f), \quad g^{(n)}(x)=f^{(n)}(x).$$

If $L(f)=0$ then $L(g)=0$ and therefore $g(b)=0$. Now it may happen that by repeated use of Rolle's theorem it can be seen by direct inspection that if $g(x_r)=0$ for all the x_r including b and perhaps for some other points, and if $g'(x_s)=0$, $g''(x_t)=0 \dots$ for all the x_s, x_t then $g^{(n)}(\xi)=0$ somewhere in the interval. If this is so then $L(f)$ must be simplex by inspection.

This is well known in the case of the Lagrange interpolation formulae and in the Taylor-Maclaurin polynomial approximation. For example, in the latter case,

$$L(f)=f(b)-f(a)-(b-a)f'(a)-\dots-\frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a).$$

The formula is of order n and if $g(b)=0=g(a)=g'(a)=\dots=g^{(n-1)}(a)$ then g' vanishes at a and at t_1 between a and b, g'' vanishes at a and at t_2 between a and t_1, and so on. Finally $g^{(n)}(\xi)=0$ at some point ξ between a and t_{n-1} and therefore between a and b.

The Euler-Maclaurin and the Gaussian quadrature formulae can be dealt with similarly. This is seen more clearly in the Gaussian formula if we write it in the form

$$L(f)=f(b)-f(a)-\Sigma B_s f'(x_s)-\Sigma 0 f(x_s)=R.$$

Thus taking three interpolation points $-t, 0, t$,

$$\begin{aligned} L(f)=f(1)-f(-1)-\tfrac{5}{9}f'(-t)-\tfrac{8}{9}f'(0)-\tfrac{5}{9}f'(t) \\ -0f(-t)-0f(0)-0f(t), \end{aligned}$$

where $t=\sqrt{(\tfrac{3}{5})}$. This is of order 7 and simplex by inspection.

$$E=L(x^7)=\tfrac{8}{25}.$$

If the end-points were (a, b) in place of $(-1, 1)$,

$$R=\frac{1}{400}\,\frac{(b-a)^7}{7!}f^{(7)}(\xi)=\frac{1}{400}\,\frac{(b-a)^7}{7!}y^{(6)}(\xi),$$

where $y=f'$ is the integrand.

Simpson's rule. Here $n=5$, and taking $a=-1$, $b=1$, $c=0$,

$$L(f)=f(1)-f(-1)-0f(0)-\tfrac{1}{3}[f'(-1)+4f'(0)+f'(1)]=R.$$

If $g(1)=g(-1)=g(0)=0$ then g' vanishes once between -1 and 0 and once between 0 and 1. Therefore g' vanishes altogether 5 times and $g^{(5)}(\xi)$ vanishes once in the interval. The formula is simplex and

$$E=L(x^5)=2-\tfrac{10}{3}=-\tfrac{4}{3}.$$

In general terms of an interval (a, b),

$$R=-\frac{\frac{4}{3}}{5!}\left(\frac{b-a}{2}\right)^5 f^{(5)}(\xi)=-\frac{1}{2880}(b-a)^5 y^{(4)}(\xi),$$

where $y=f'$ is the integrand.

We can invent other simplex quadrature formulae.

Closed Gaussian formula. This is similar to the Gaussian but using the end-points in addition to internal points. The latter will be roots of the equation $P_r'(x)=0$ where P_r' is the derivative of a Legendre polynomial. We state the case with two interior points.

$$L(f)=f(1)-f(-1)-\tfrac{1}{6}[f'(-1)+5f'(-t)+5f'(t)+f'(1)]$$
$$-0f(-t)-0f(t). \quad \text{......(3)}$$

Here $t^2=\tfrac{1}{5}$, $n=7$ and the formula is simplex by inspection,

$$E=-\tfrac{32}{75}=-\tfrac{1}{300}(b-a)^7.$$

Mixed Simpson-Euler-Maclaurin formula. This differs from the Euler-Maclaurin in including the value of the integrand at the mid-point. For a single pair of intervals going as far as $f''(x)$,

$$L(f)=f(1)-f(-1)-\tfrac{1}{2}[f'(-1)+2f'(0)+f'(1)]$$
$$+\tfrac{1}{15}[\tfrac{1}{2}f'(-1)+\tfrac{1}{2}f'(1)-f'(0)+f''(1)-f''(-1)], \text{...(4)}$$

to which we can add a term $0f(0)$. Here $n=7$, the formula is simplex by inspection and

$$E=\frac{16}{15}\left(\frac{b-a}{2}\right)^7=\frac{1}{120}(b-a)^7.$$

Besides those given above there are other formulae which can be easily seen to be simplex. For example, the Laplace forward or backward difference formulae. To take an instance :

$$L(f)=f(1)-f(0)-\tfrac{1}{2}[f'(0)+f'(1)]+\tfrac{1}{12}\Delta^2 f'(0)-\tfrac{1}{24}\Delta^3 f'(0)$$
$$=f(1)-f(0)-\tfrac{3}{8}f'(0)-\tfrac{19}{24}f'(1)+\tfrac{5}{24}f'(2)-\tfrac{1}{24}f'(3). \quad \text{......(5)}$$

Here $n=5$ and

$$E=-\frac{19}{6}=-\frac{19}{18\times 81}(b-a)^5,$$

taking $a=0$, $b=3$.

Another example is obviously that of various orders of derivatives in terms of forward or backward differences. But a more interesting case is that in which central differences are used.* These are given

* L. M. Milne-Thomson, *Calculus of Finite Differences*, p. 160.

by Milne-Thomson with complicated remainders obtained by the process of reasoning used. The formulae are however simplex and only the first term in the remainder is necessary. We take as an example the case of seven interpolation points and modify his notation using x in place of his y and 0 in place of his x. We also take $\omega = 1$ for simplicity. For the third and fourth derivatives, we have

$$L_3(f) = f^{(3)}(0) - \mu\delta^3 u_0 + \tfrac{1}{4}\mu\delta^5 u_0 = R_3,$$
$$L_4(f) = f^{(4)}(0) - \delta^4 u_0 + \tfrac{1}{6}\delta^6 u_0 = R_4.$$

In the former case first suppose f to be odd. Then subtracting an odd polynomial of degree 5 we can make $g = f - p = 0$ at $x = 0, \pm 1, \pm 2, \pm 3$.

When this has been done, $g'' = 0$ at 0 and at 4 other points. Hence $g''' = 0$ at 4 points distinct from $x = 0$. But if $L_3(g) = 0$ then $g''' = 0$ also at $x = 0$. Therefore g''' vanishes at 5 points and $g^{(7)}(x)$ vanishes once. The same will be true for $f(x)$. If $f(x)$ is not odd it can be expressed as $h(x) + l(x)$ where $h(x)$ is odd and $l(x)$ even. Then $L(f) = L(h) = 0$ and $h^{(7)}(t) = 0$ at some point t. If $t = 0$, $l^{(7)}(t) = 0$ and $f^{(7)}(t) = 0$. If $t \neq 0$, $h^{(7)}(-t) = h^{(7)}(t) = 0$ and $l^{(7)}(-t)$, $l^{(7)}(t)$ are of opposite sign. Therefore $f^{(7)}(-t)$, $f^{(7)}(t)$ are of opposite sign and $f^{(7)}(\xi) = 0$ somewhere between. This shows that $L_3(f)$ is simplex.

Similarly by choosing first an even function we could show that $L_4(f)$ is simplex. To find E_3 choose

$$p_n(x) = x(x^2-1)(x^2-4)(x^2-9).$$

Then $$E_3 = L_3(p_n) = p_n{}^{(3)}(0) = 49 \times 6.$$

Hence $$L_3(f) = \frac{49 \times 6}{7!} f^{(7)}(\xi) = \frac{7}{120} f^{(7)}(\xi).$$

For E_4 choose $$p = x^2(x^2-1)(x^2-4)(x^2-9).$$

$$E_4 = L_4(p) = p^{(4)}(0) = 49 \times 24.$$

$$L_4(f) = \frac{49 \times 24}{8!} f^{(8)}(\xi) = \frac{7}{240} f^{(8)}(\xi).$$

Addition of Formulae. If $L(f)$ is the sum of two or more simplex formulae of the same order and if the corresponding E's for these are *all of the same sign*, then $L(f)$ is simplex. For let

$$L(f) = L_1(f) + L_2(f) + L_3(f)$$
$$= \frac{E_1}{n!} f^{(n)}(\xi_1) + \frac{E_2}{n!} f^{(n)}(\xi_2) + \frac{E_3}{n!} f^{(n)}(\xi_3).$$

If $L(f) = 0$ since E_1, E_2, E_3 are of the same sign either all the $f^{(n)}(\xi)$ vanish or at least two are of opposite sign. Now $f^{(n)}(x)$ is the derivative of $f^{(n-1)}(x)$ and therefore if it is opposite signs at two values of x it vanishes somewhere between. This proves our statement.

All the quadrature formulae are used with many intervals and thus if they are simplex for one (or two intervals) they are also simplex for r (or $2r$) intervals by simple addition.

Mixed Simpson-Euler-Maclaurin formula for 2r intervals. This lends itself readily to computation. Let $b-a=2rh$,

$$y_s=y(a+sh)=f'(a+sh), \quad m=\tfrac{1}{2}(y_0+y_{2r}).$$

Then from (4),

$$f(b)-f(a)=\int_a^b y\,dx=h(m+y_1+y_2+\dots+y_{2r-1})$$

$$+\frac{h}{15}[(y_1-m)+(y_3-y_2)+\dots+(y_{2r-1}-y_{2r-2})-h(y'_{2r}-y'_0)]+R. \quad (6)$$

The column of m, y_1, y_2, ... is first written and then, in a second column, the *alternate* differences are written and the columns added.

For example, if $a=1$, $b=2$, $f'(x)=y=1/x$, $2r=10$, $h=0{\cdot}1$, the remainder is

$$R=5\times\frac{16}{15}\times\frac{10^{-7}}{7!}\,y^{(6)}(\xi)$$

$$=\tfrac{16}{21}\theta\times10^{-7},$$

where θ lies between 2^{-7} and 1. To nine places the computation gives

$$0{\cdot}69314{,}7169.$$

The value of log 2 correct to eleven places * is

$$\log 2=0{\cdot}69314{,}71805{,}6.$$

The next more accurate Simpson-Euler-Maclaurin formula would be

$$\int_a^b y\,dx=h(m+y_1+\dots+y_{2r-1})$$

$$+\frac{h}{63}[(y_1-m)+(y_3-y_2)+\dots+(y_{2r-1}-y_{2r-2})$$

$$-5h(y'_{2r}-y'_0)+\tfrac{1}{15}h^3(y'''_{2r}-y'''_0)]+R, \quad \dots(7)$$

and for the same case as above the calculation, to eleven places, gives

$$0{\cdot}69314{,}71806{,}7.$$

The remainder, theoretically, is

$$R=5\times\left(-\frac{64}{35}\right)\times\frac{10^{-9}}{9!}\times8!\,\theta$$

$$=-1{\cdot}02\times10^{-9}\theta \qquad (2^{-9}<\theta<1).$$

Three-eighths rule. Here

$$L(f)=f(3)-f(0)-\tfrac{3}{8}[f'(0)+3f'(1)+3f'(2)+f'(3)].$$

* J. C. Adams, *Collected Scientific Papers I*, p. 464.

If $L_1(f)$ is the Laplace forward formula (5) given above for $f(1)-f(0)$, and if

$$L_2(f)=f(3)-f(1)-\tfrac{1}{3}[f'(1)+4f'(2)+f'(3)],$$

the Simpson's rule from $x=1$ to $x=3$, then if $a=0$, $b=3$,

$$E_1=-\frac{19}{18\times 81}(b-a)^5,\quad E_2=-\frac{8}{18\times 81}(b-a)^5.$$

These are of the same sign and the three-eighths rule

$$L(f)=L_1(f)+L_2(f)$$

is simplex. For this rule

$$E=E_1+E_2=-\tfrac{1}{54}(b-a)^5,$$

$$L(f)=-\frac{1}{54\times 120}(b-a)^5f^{(5)}(\xi)=-1{\cdot}54\times 10^{-4}(b-a)^5y^{(4)}(\xi).$$

The Cotes' formula with an odd number of intervals, of which the above rule is one, can be seen to be simplex by adding a suitable Laplace forward formula to the Cotes' formula with one fewer interval. But to see that the formula with an even number of intervals is simplex requires a more complicated technique.

Illustration. Finally we give an example which is of no utility but which illustrates some important points. Let $-1<a<b<1$. By Lagrange we can express $f(a)$, $f(b)$ in terms of $f(-1)$, $f(0)$, $f(1)$. Let

$$L(f)=f(a)+f(b)-k_1f(-1)-k_2f(0)-k_3f(1)=R.$$

The order of $L(f)$ is 3 unless $b=-a$ or $a^2-ab+b^2=1$. If a, b both lie in (0, 1) or in $(-1, 0)$ the formula is clearly simplex since the Lagrange formulae have E's of the same sign. If a, b are in different intervals the E's are of opposite sign and the method has to be modified. We express $f(b)$ in terms first of $f(-1)$, $f(a)$, $f(0)$ and then in terms of $f(a)$, $f(0)$, $f(1)$. We have

$$L_1(f)=f(b)-\frac{b(1+b)}{a(1+a)}f(a)-Af(-1)-Bf(0),$$

$$L_2(f)=f(b)-\frac{b(1-b)}{a(1-a)}f(a)-Cf(0)-Df(1).$$

For these two formulae

$$E_1=b(b+1)(b-a),\quad E_2=-b(1-b)(b-a)$$

and are of opposite sign. If it is possible to choose w_1, w_2 of the same sign and such that $w_1-w_2=1$ and

$$w_2\frac{b(1-b)}{a(1-a)}-w_1\frac{b(1+b)}{a(1+a)}=1,$$

then
$$\begin{aligned}L(f)&=w_1L_1(f)-w_2L_2(f)\\&=f(b)+f(a)-k_1f(-1)-k_2f(0)-k_3f(1)\end{aligned}$$

will be simplex of order 3 with the proper choice of k_1, k_2, k_3. A little algebra shows that our formula is simplex of order 3 if

$$(a+b) \pm (a^2+b^2)$$

are of the same sign. If we mark points in a plane with coordinates (a, b) the condition is that (a, b) lies in one of the two circles circumscribing the unit squares in the first and third quadrants.

This is an example which might be called " excessive " in that it is not in general possible to find a polynomial, p, of degree $3-1$, so as to make $g=f-p$ vanish at all interpolation points other than b. Quadrature formulae such as Simpson's rule with $2r$ intervals $(r>1)$ are similarly excessive.

Our illustration can also be regarded as the solution of the following problem: If $-1<a<b<1$, under what conditions does the vanishing of $f(x)$ at -1, 0, 1 together with the vanishing of $f(a)+f(b)$ imply the vanishing of $f^{(3)}(x)$ somewhere between -1 and $+1$? P. J. D.

1322. Giles Hussey (1710-1788) . . . was born a gentleman and studied in Italy on an allowance from his father. He recognised that the boundless esteem in which Graeco-Roman statues were then held had set up bad standards and noxious prejudices as a standard of beauty; and he came to the conclusion that " ideal form " must be mathematically evolved. He tried to work out a theory of " Beauty " on a mathematical basis, and talked so much about it to his friends that they all regarded him as mad. As he showed no signs of being able to make money by his art his father stopped his allowance; and thereafter he seems to have had an unsuccessful career in spite of help and encouragement from various patrons when he returned to England. . . . Very few of his works seem to have survived. The print room of the British Museum has some drawings, including a red chalk head and shoulders of a young man executed with great precision of hand and very deliberate stylisation, and a curious profile drawn with pronounced use of semicircular lines, said to be from an antique gem. It is clear that both as a theoretician and as an artist Hussey belongs more to the twentieth century than his own.

The same applies to Francis Towne (1740-1816), whose drawings in intention are strangely close to certain aspects of twentieth-century art. . . . In Switzerland he was interested both in the geological and the architectural structure of the mountains. Some of his drawings are expressed in a language that is as deliberately symbolic and geometrical as the *Still Life* by Paul Nash. —Wilenski, *English Painting*, pp. 97, 98. [Per Mr. E. H. Lockwood.]

1323. Were we to revert to the system of expressing points obtained as a percentage of possible points Mr. Whatley's scheme would encourage unfinished matches, for *five points out of five gives almost as high a percentage as twenty points out of twenty.*—Letter to *The Times*, 15th August, 1939. [Per Mr. F. W. Kellaway.]

1324. I consider that the surface of the second degree at present, whatever may be the case in some future development, stands on a platform of its own on account of the services which it has rendered to all departments of Mathematical Science, and well deserves a distinctive name instead of being recognised only by its number, a mode of designation which, I am informed, a convict feels so acutely.—Percival Frost, Preface to *Solid Geometry* (1875). [Per Mr. F. W. Kellaway.]

ROTATION AND STRAIN.

BY K. E. BULLEN.

1. Introduction to the principles of elasticity is often received by the average student in a rather piecemeal fashion, the ideas of strain, shear, etc., being indicated from consideration of a number of very special cases. There is a danger that the student may not see the forest for the trees for a considerable time, and a more general approach to the analysis of strain, if not too difficult, would have obvious advantages. The general presentation as in Love's *Elasticity* is probably too lengthy for a first approach to the subject, and the following method is suggested as being perhaps suitable for the purpose. Any difficulty that a junior student might find with the mathematics used is compensated by the extreme conciseness of the method. Moreover, the mathematical methods used are all important ones, and the treatment might well serve also as a first introduction to some of these methods. The incidental derivation of the rotation formula would have the further advantage that the student would be shown at an early stage the significance of the curl of a vector.

The earlier part of the treatment, associated with the splitting up of the strain tensor into its symmetrical and antisymmetrical parts is carried out with the use of the summation convention after the manner in Jeffreys' *Cartesian Tensors*; it is then shown that expressions for rotation and shear may be deduced by fairly direct methods.

2. Suppose that a piece of elastic matter has undergone a change in configuration so that the displacement of a typical particle P at the point x_i has been u_i $(i=1, 2, 3)$. Let Q be a neighbouring particle at the point x_i+y_i. Then if Q be sufficiently close to P its displacement will have been $u_i+(\partial u_i/\partial x_j)\,y_j$.

The displacement of Q relative to P, *i.e.* $\dfrac{\partial u_i}{\partial x_j}y_j$, may be put in the form

$$e_{ij}y_j-\xi_{ij}y_j, \qquad \text{.............................(1)}$$

where

$$\xi_{ij}=\frac{1}{2}\left(\frac{\partial u_j}{\partial x_i}-\frac{\partial u_i}{\partial x_j}\right) \qquad \text{.............................(2)}$$

and

$$e_{ij}=\frac{1}{2}\left(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}\right). \qquad \text{.............................(3)}$$

In the normal problems that arise, the relative displacements involved are so small that squares and products of the elements of $\partial u_i/\partial x_j$ may be neglected. Thus in the subsequent analysis it will be permissible to consider the effects of separate members of (2) and (3), and then superpose the results.

On transforming axes by the usual methods, it is seen that ξ_{ij} and e_{ij} obey the transformation laws for tensors of the second order. Moreover ξ_{ij} is antisymmetrical and e_{ij} symmetrical.

3. Consider first ξ_{ij}. The contribution to the relative displacement (1) due to the particular pair of elements ξ_{23} and ξ_{32} of (2) has components $(0, -\xi_{23}y_3, \xi_{23}y_2)$. The associated alteration to the magnitude of the distance between P and Q will therefore have been

$$\{y_1{}^2 + (y_2 - \xi_{23}y_3)^2 + (y_3 + \xi_{23}y_2)^2\}^{\frac{1}{2}} - \{y_1{}^2 + y_2{}^2 + y_3{}^2\}^{\frac{1}{2}}, \quad \text{.........(4)}$$

which is zero to the first order in ξ_{23}. The direction of PQ will have been turned through an angle

$$\tan^{-1}\left(\frac{y_3 + \xi_{23}y_2}{y_2 - \xi_{23}y_3}\right) - \tan^{-1}\left(\frac{y_3}{y_2}\right). \quad \text{..................(5)}$$

The latter expression is equal to $\tan^{-1}\xi_{23}$, which to the first order reduces to ξ_{23}, and is incidentally independent of y_2, y_3. The pair of elements ξ_{23} and ξ_{32} is thus associated with a rotation as of a rigid body through an angle ξ_{23} about the 1-axis; and similarly for the other two pairs of non-zero elements of ξ_{ij}. The complete tensor ξ_{ij} thus corresponds to a pure rotation about an axis through P, and so is called the rotation tensor at P. The rotation may also be regarded as a vector of components $(\xi_{23}, \xi_{31}, \xi_{12})$; the vector whose components are double the latter, *i.e.*

$$\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \ \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \ \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)$$

is called the "curl" or "rot" of the vector u_i.

4. The tensor e_{ij} is called the strain tensor at P. The contribution to (1) due to the element e_{11} has components $(e_{11}y_1, 0, 0)$. This clearly constitutes a strain in which all small lengths in the direction of the 1-axis have been uniformly increased in the ratio e_{11}. Such a strain is an extension. The elements e_{11}, e_{22}, e_{33} thus give three mutually perpendicular extensions in the directions of the axes.

5. The contribution to (1) due to the pair of elements e_{23} and e_{32} has components $(0, e_{23}y_3, e_{23}y_2)$. If expressions be formed analogous to (4) and (5), it will be found that the join of P and Q has suffered a change in both magnitude and direction. The change in magnitude is seen to be proportional to y_2y_3, while the angle of rotation about the 1-axis reduces to

$$e_{23} \cdot \left(\frac{y_2{}^2 - y_3{}^2}{y_3{}^2 + y_3{}^2}\right). \quad \text{........................(6)}$$

The latter, unlike ξ_{23}, depends on the direction of PQ. If R and S be two particles near P on the 2- and 3-axes respectively, the expression (6) takes the values e_{23} and $-e_{23}$. Thus the displacement associated with e_{23} and e_{32} will have been an alteration in the angle RPS of amount $2e_{23}$. Such a deformation is a shear. Similar results hold for the remaining two pairs of elements of e_{ij}. The three independent members of $2e_{ij}(i \neq j)$ are called shearing strains.

6. By introducing the strain quadric $e_{ij}y_iy_j = \text{constant}$, the principal axes of strain may then be introduced and the usual theory developed.

K. E. B.

SOME MISSING FIGURE PROBLEMS AND CODED SUMS.

By C. Dudley Langford.

It sometimes happens that teachers want missing figure problems and coded sums, for instance, at the end of term. These given here may be of use in saving teachers from having to make them up themselves. To save space, the missing figure problems are given as a whole, with the data printed in clarendon type. In setting these problems to a class the exact position of *every* missing digit must, of course, be indicated : this applies to remainders as well as to the main sum.

```
9 ) 78935              8 ) 2911            11 ) 1090267
    8770   rem. 5           363   rem. 7         99115   rem. 2
```

```
 7 | 37037                      12 | 256499
 9 |  5291                       9 |  21374   r. 11
       587   r. 8                      2374   r. 8
   rem. 56                         rem. 107
```

```
 6 | 746057                     11 | 128195
 8 | 124342   r. 5              12 |  11654   r. 1
      15542   r. 6                      971   r. 2
   rem. 41                         rem. 23
```

```
 5 | 247891                     12 | 71047
 7 |  49585   r. 1               7 |  5920   r. 7
11 |   7083   r. 4               2 |   845   r. 5
        643   r. 10                    422   r. 1
   rem. 371                        rem. 151
```

In the two following sums, the divisors, dividends and remainders are the same, the last being **100**.

```
 3 | 1885                        7 | 1885
 5 |  628   r. 1                 5 |  269   r. 2
 7 |  125   r. 3                 3 |   53   r. 4
       17   r. 6                       17   r. 2
```

```
        6021                          9019
37 ) 222778                    91 ) 8200009
     222                             819
       77                             100
       74                              91
        38                             909
        37                             819
         1                              90
```

```
       67                         1103
47 ) 3189                    36 ) 39724
     282                          36
      369                          37
      329                          36
       40                           124
                                    108
                                     16

         72194                     2362
417 ) 30114114              347 ) 819765
      2919                        694
        924                       1257
        834                       1041
         801                       2166
         417                       2082
         3841                        845
         3755                        694
           884                       151
           834
            50

        4472                        459
659 ) 2947654               321 ) 147652
      2636                        1284
       3116                        1925
       2636                        1605
        4805                        3202
        4613                        2889
         1924                        313
         1318
          606

       433                         674
149 ) 64624                 239 ) 161313
      596                         1434
       502                         1791
       447                         1673
        554                         1183
        447                          956
        107                          227

      41347                        7942
      67996                        6873
      97637                       51294
      34195                        6079
      85768                       18356
     323943                       90544
```

```
          5739
874 ) 5016279
      4370
       6462
       6118
        3447
        2622
         8259
         7866
          393
```

```
          1248
841 ) 1050137
      841
      2091
      1682
        4093
        3364
         7297
         6728
          579
```

```
100243
  9976
------
 90276
```

```
100294
   708
------
 99586
```

```
450640
449705
------
   935
```

```
 84365
     9
------
759285
```

```
201165
107969
------
 93196
```

```
 73049
     7
------
511343
```

```
  8493
    12
------
101916
```

```
 37032
    27
------
259259
74074
------
999999
```

Note: the multiplier is a multiple of three.

```
   4736
    409
-------
  42624
18944
-------
1937024
```

```
  76054
     81
-------
  76054
608432
-------
6160374
```

The multiplier is a multiple of 9, the multiplicand a multiple of 11.

```
    41586
     7009
---------
   291102
   374274
---------
291476274
```

```
   6377
    713
-------
  19131
  6377
44639
-------
4546801
```

```
   9435
    922
-------
  18870
 18870
84915
-------
8699070
```

```
   8247
    923
-------
  24741
 16494
74223
-------
7611981
```

```
   423
   891
------
   423
 3807
3384
------
376893
```

R

Contracted methods of multiplication are used in the next three sums. In the two right-hand columns of the first sum the same figure has to be supplied six times.

```
   98476            46237
     366              749
  ------          -------
  590856          323659
 3545136           2265613
 --------         --------
 36042216         34631513
```

```
   57601
    8412
  ------
  691212
 4838484
 ---------
 484539612
```

The following contracted multiplication sum has the two solutions here printed ; the pupil should be told this and set to find both.

```
    xxxxx
     xxxx
   ------
   445428
 xxxxxxx
 ---------
 401330628
```

```
    74238           49492
     5406            8109
   ------          ------
   445428          445428
 4008852         4008852
 ---------       ---------
 401330628       401330628
```

Here is another with two solutions. The pupil should be told this, and also that, though the method of solution is simple in practice, it requires patience.

```
    xxxxx
      xxx
   ------
   xxxxxx
 xxxxxx
 --------
 22555533
```

```
   55419            37037
     407              609
   ------           ------
   337933           333333
 221676           222222
 --------         --------
 22555533         22555533
```

CODE SUMS

$$\begin{array}{rcr} BA & & 83 \\ ES & & 64 \\ \cline{1-1}\cline{3-3} SXB & \text{Multiplication} & 498 \\ AAK & & 332 \\ \cline{1-1}\cline{3-3} RAVK & & 5312 \\ END & \text{Addition} & 678 \\ \cline{1-1}\cline{3-3} RXXD & & 5990 \\ ADE & \text{Subtraction} & 306 \\ \cline{1-1}\cline{3-3} REBS & & 5684 \end{array}$$

Finding prime factors :

$$\begin{array}{r|r} A & BCCB \\ BB & BDE \\ BE & BE \\ & B \end{array} \qquad \begin{array}{r|r} 7 & 1001 \\ 11 & 143 \\ 13 & 13 \\ & 1 \end{array}$$

$$\begin{array}{r|r} A & BCBD \\ A & DDBE \\ DE & DFG \\ DE & DE \\ & D \end{array} \qquad \begin{array}{r|r} 7 & 8281 \\ 7 & 1183 \\ 13 & 169 \\ 13 & 13 \\ & 1 \end{array}$$

Hints which will be useful in the following multiplication sums :

$$\begin{array}{llll} AB \times AB & CAB\,; & 25 \times 25 & 625\,; \\ AA \times AA & ABA\,; & 11 \times 11 & 121\,; \\ AB \times AB & ACC\,; & 12 \times 12 & 144\,; \\ \left\{\begin{array}{l} AB \times AB \\ BA \times BA \end{array}\right. & \begin{array}{l} ACD\,; \\ DCA\,; \end{array} & \left\{\begin{array}{l} 13 \times 13 \\ 31 \times 31 \end{array}\right. & \begin{array}{l} 169\,; \\ 961. \end{array} \end{array}$$

$$\begin{array}{r} AB \\ CB \\ \hline AB \\ ACC \\ \hline CEDC \end{array} \qquad \begin{array}{r} 79 \\ 19 \\ \hline 79 \\ 711 \\ \hline 1501 \end{array}$$

$$\begin{array}{r} AA \\ AA \\ \hline BCD \\ BCD \\ \hline BBDD \end{array} \qquad \begin{array}{r} 88 \\ 88 \\ \hline 704 \\ 704 \\ \hline 7744 \end{array}$$

$$\begin{array}{r} AB \\ AB \\ \hline CA \\ AB \\ \hline CDB \end{array} \qquad \begin{array}{r} DC \\ DC \\ \hline EBE \\ DFB \\ \hline EGDB \end{array} \qquad \begin{array}{r} 31 \\ 31 \\ \hline 93 \\ 31 \\ \hline 961 \end{array} \qquad \begin{array}{r} 69 \\ 69 \\ \hline 414 \\ 621 \\ \hline 4761 \end{array}$$

Finding $(AB)^4$

AB	32
AB	32
CD	96
DE	64
$GFBE$	1024
$GFBE$	1024
$GFBE$	1024
$BFEH$	2048
$EFCD$	4096
$GFEHJKD$	1048576

ABC	276
DBC	476
$EFED$	1104
$FGHA$	1932
$FCJC$	1656
$FHFHBC$	131376

ABB	255
CBB	355
EDB	765
$FAEB$	1275
$FAEB$	1275
$HGBAB$	90525

ABC	125
DBC	325
DEC	375
BCF	250
GBC	625
$HFGBC$	40625

ABB	366
CBB	566
$FEAD$	1830
$GFHB$	2196
$GFHB$	2196
$GHJFCB$	207156

ABC	726
CDE	648
$DGFC$	4356
$BHJD$	2904
$FEJE$	5808
$DAJDDE$	470448

ABC	712
DBC	512
GDFE	3560
ABC	712
BHCH	1424
GFHDHH	364544
ABCA	1091
DCE	495
DFED	4364
CGAC	9819
EDEE	5455
EDBBDE	540045

C. D. L.

1325. Another odd scientific discovery was made in the course of war work in British Government laboratories at Teddington, near London. It was then found that, for the diameter of an object to measure exactly the same in every direction, the object need not necessarily be a perfect globe. When roller bearings were being examined at the laboratory, a man rolled a short stick of clay evenly between two boards, a fixed distance apart. The clay did not emerge in circular form, as expected, but triangular. This discovery showed that a three-, five-, or seven-sided figure might have the same diameter in every direction, and that a cylinder of steel need not be perfectly circular. A Lancashire tool and gear consultant working on this accidental discovery has since found that an angular solid, as well as a globe, can have the same diameter in every direction. So the practical engineer found that roller-bearings need not be circular.—H. T. Wilkins, *Mysteries of the Great War*, p. 18. [Per Mr. F. R. Curtis.]

1326. It is highly desirable that Teachers and Practical Men should possess some knowledge of this most important branch of pure mathematics, in order to enable them to understand our best works on mechanical and experimental philosophy. The great physical laws, by which it has pleased the Almighty to govern the universe, must always form a lofty subject of contemplation to his intelligent creatures; but these laws can only be duly interpreted by the aid of the symbolic language of the higher analysis.—Thomas Tate, Preface to *The Principles of the Differential and Integral Calculus Simplified and Applied to the Solution of Various Useful Problems in Practical Mathematics and Mechanics* (1849). [Per Mr. F. W. Kellaway.]

1327. . . . To assert that the " forces " of an organism must necessarily, or indeed probably, be physico-chemical forces is sillier still. We know nothing whatever about " force " or " forces ". The only thing we can truly be said to know is rate of change of (relative) position. " Force " is a convenient concept. What force really is, or indeed whether there is any such thing, is a matter on which we know nothing. Surely, then, to assert dogmatically that all " forces ", gravitational, physical, chemical, etc., must be of one and the same nature, and capable of expression by one and the same laws, is the height of folly. We have all heard of the blind man, in a dark cellar, at midnight, looking for a black cat which isn't there. Here we see, to have a blind man, in a dark cellar, at midnight, loudly asserting, of a number of black cats which aren't there, that they are all of the same breed.—Canon Peter Green, *The Problem of Right Conduct*, pp. 43-44. [Per Mr. A. F. Mackenzie.]

AN INVESTIGATION INTO MULTIPLICATION. II. LONG MULTIPLICATION OF MONEY.

By H. Webb.

Introduction.

Here again we are confronted with a variety of methods. For purposes of work in the early stages at least two of these are worthy of consideration. They are the factorial or Ten-ten and the wholesale methods. This latter is sometimes called the "Ballard Box Method" although it was known and used (but not extensively) long before Dr. Ballard's work on this subject.

The Ten-ten Method.

In questions such as £56 17s. $8\frac{3}{4}$d. $\times 37$ some teachers instruct their pupils to factorise the multiplier and so we have

(i) $6 \times 6 + 1$ or (ii) $4 \times 9 + 1$ or (iii) $3 \times 12 + 1$,

with the whole sum set out as shown below:

£	s.	d.
56	17	$8\frac{3}{4}$
		6
341	6	$4\frac{1}{2}$
		6
2047	18	3
56	17	$8\frac{3}{4}$
2104	15	$11\frac{3}{4}$

The varied selection of factors makes it difficult for the teacher to check the pupils' work. Further, the factorisation of a number such as 249 is altogether too cumbersome for young pupils. For these reasons teachers more often resort to the Ten-ten method, which appears in three forms:

E.g. £78 16s. $9\frac{3}{4}$d. $\times 249$ results in:

	£	s.	d.	
(*a*)	78	16	$9\frac{3}{4}$	$\times 9$
			10	
	788	8	$1\frac{1}{2}$	$\times 4$
			10	
	7884	1	3	
			2	
	15768	2	6	(=200 times)
	3153	12	6	(= 40 ,,)
	709	11	$3\frac{3}{4}$	(= 9 ,,)
	19631	6	$3\frac{3}{4}$	

N.B.—It is not necessary to enter the items which are in brackets.

$$
\begin{array}{lrrrcrrr}
 & \text{£} & \text{s.} & \text{d.} & & \text{£} & \text{s.} & \text{d.} \\
(b) & 78 & 16 & 9\frac{3}{4} & \times 9 = & 709 & 11 & 3\frac{3}{4} \\
 & & & 10 & & & & \\
\hline
 & 788 & 8 & 1\frac{1}{2} & \times 4 = & 3153 & 12 & 6 \\
 & & & 10 & & & & \\
\hline
 & 7884 & 1 & 3 & \times 2 = & 15768 & 2 & 6 \\
\hline
 & & & & & 19631 & 6 & 3\frac{3}{4} \\
\hline
\end{array}
$$

$$
\begin{array}{lrrrcrrr}
 & \text{£} & \text{s.} & \text{d.} & & \text{£} & \text{s.} & \text{d.} \\
(c) & 78 & 16 & 9\frac{3}{4} & & 709 & 11 & 3\frac{3}{4} \\
 & & & 10 & & 3153 & 12 & 6 \\
\hline
 & 788 & 8 & 1\frac{1}{2} & & 15768 & 2 & 6 \\
\hline
 & & & 10 & & 19631 & 6 & 3\frac{3}{4} \\
\hline
 & 7884 & 1 & 3 & & & & \\
\hline
\end{array}
$$

This Ten-ten method has an advantage in that multiplication by 10 is an easy operation, but the division by 12 and 20 to reduce pence to shillings and shillings to pounds alternating with multiplication by 10 produces many mistakes.

In form (*a*) the sums become long and straggling with large multipliers; in (*b*) the spaces between the final partial products often cause bad alignment which leads to errors. Further there are certain forms of arranging the working of sums with which the pupil is always more comfortable and certainly more sure. From the results of my tests on formal arrangement (pp. 266-267) I found that in questions such as: £7 16s. $9\frac{3}{4}$d. $\times$ 6, form (i) below, was 8 per cent. faster and 8 per cent. more accurate than form (ii):

$$
\begin{array}{lrrr}
 & \text{£} & \text{s.} & \text{d.} \\
\text{(i)} & 7 & 16 & 9\frac{3}{4} \\
 & & & 6 \\
\hline
 & 47 & 0 & 10\frac{1}{2} \\
\hline
\end{array}
\qquad
\text{(ii) £7 16s. } 9\tfrac{3}{4}\text{d.} \times 6 = \underline{\text{£47 0s. } 10\tfrac{1}{2}\text{d.}}
$$

It follows from this that the departure from the usual formal arrangement, in obtaining the final partial products in the above (*b*) and (*c*) forms of the Ten-ten method, results in a loss in accuracy and time.

THE WHOLESALE METHOD.

This is illustrated in the model below:

£	s.	d.	
78	16	$9\frac{3}{4}$	
		249	
19631	6	$3\frac{3}{4}$	f.
209	202	186	4) 747
17430	2490	2241	$186\frac{3}{4}$
1992	1494	12) 2427	
19631	$2\cancel{0}$) $418\cancel{6}$	202 r. 3	
	209		

I have known parents (themselves brought up on the Ten-ten method) horrified at finding their children using the Wholesale method. "Multiplying a large sum of money by such a number as 249 all in one straight line is too much of a good thing", they have told me. At first glance this method certainly does appear to be cumbersome. A little further consideration, however, will rob it of most of its fearsomeness. On examination it will be noticed that the 249 becomes the multiplicand which is multiplied by each item, in turn, of the sum of money. Thus the sum becomes very little more than a progressive form of short multiplication of number.

Dr. Ballard * has proved this method to be quicker and more accurate than the Ten-ten method, and my own observations from tests † that I have given support his findings.

It must not be thought, however, that the Wholesale method is without drawbacks. It possesses several disadvantages. In the first place it is necessary to space widely the pounds, shillings, and pence—they should be spaced almost (but not quite) right across the page of the average exercise book. In the second place the "working out" figures below the space for the final answer tend to become small and untidy—a kind of marginal scribbling. Untidy work is fatal to accuracy—all work MUST be neat and orderly.

* *Teaching the Essentials of Arithmetic* (Chap. XIII), Ballard.

† Tests that I gave in 1936 when preparing my *Arithmetic in Daily Life* (Cambridge University Press). I possess no record of these tests, so that recently I asked one of my best teachers to carry out further tests in an endeavour to obtain some results. She commenced with the top class in the school, but the children working according to the Ten-ten method were, comparatively, so extremely slow and obtained such outstandingly bad results that the test was abandoned. The teachers felt that its continuance would be a serious hindrance to the work in arithmetic.

I tested myself with both methods by working four questions of the order £76 18s. $11\frac{3}{4}$d. $\times$ 179 in the following rotation:

(i) Ten-ten method.
(ii) Wholesale method.
(iii) Wholesale method.
(iv) Ten-ten method.

In each case with the Ten-ten method it took me two minutes to obtain the answer. With the Wholesale method it took me 1 min. 42 sec. and 1 min. 37 sec. respectively.

The " working out " figures should be just as large and equally as neat as the figures of the answer. And, finally, I have discovered that in a number of cases the fact that the multiplicand lies beneath the multiplying figures causes a certain amount of confusion.

£	s.	d.	
31	17	$9\frac{3}{4}$	
		56	
1785	17	6	f.
49	45	42	4) 168
155	85	504	42d.
186	102	12) 546	
1785	2Ø) 997	45s. 6d.	
	£49 17s.		

Here is an example of this confusion. It will be noticed that the pupil has treated the farthings and pence as multipliers but has regarded the shillings and pounds as multiplicands. Illustrations of this will be found on pp. 261-2 showing one of the actual papers submitted by a pupil during the tests mentioned on pp. 257-8.

The first two objections to the Wholesale method, pp. 255-6, can be overcome satisfactorily with reasonable drill. The last difficulty can be removed by the simple expedient of placing the multiplier (really the multiplicand) above the sum of money thus :

		249	
£78	16	$9\frac{3}{4}$	
19631	6	$3\frac{3}{4}$	f.
209	202	186	4) 747
1743	249	2241	186d. 3 f.
1992	1494	12) 2427	
19631	2Ø) 418$\cancel{6}$	202s. 3d.	
	£209 6s.		

In setting out the working in this form the pupil must be instructed to avoid placing the £ s. d. signs above the monetary quantities. All that is necessary is to place the £ mark immediately to the left of the pounds' figures.

I have found this form possesses an additional advantage over that more commonly used (see pp. 255-6).* The " working out " figures below are closer to the £78 16s. $9\frac{3}{4}$d., and the general neatness of arrangement and subsequent accuracy seem to be strengthened by the better alignment which creeps in between these two quantities.

Preliminary Tests in Long Multiplication of Money.

In my Senior School which uses the Wholesale method I tested

* In the following pages this will be referred to as the Normal Method, and the second example above as the Suggested Method.

every pupil, with the exception of those in a backward class, with both the Normal and the Suggested methods. Each class was divided into two sections as nearly as possible equal to each other in arithmetical ability. One half was given a test of five sums to work according to the Normal method and the other half a similar test to work according to the Suggested method. This constituted Part I of the preliminary tests.

Some time later—in all cases but one, at the same hour (9.30 a.m.) two days later—Part II of the preliminary tests was given. In this the pupils who had previously worked according to the Normal now used the Suggested method. Those who had used the Suggested method now changed to the Normal. Thus Part II of the tests was a reversal of Part I.

The Results of the Preliminary Tests

The results of the tests indicated that the Suggested method was an improvement on the Normal. The combined results of Parts I and II tests (see page 263) show the Suggested method to be 4 per cent. more accurate and 3 per cent. quicker than the Normal.

A Pupil's Part I and Part II Tests Papers.

On pages 260-262 I have included the actual Parts I and II test papers submitted by one of the pupils. The Part II paper, worked according to the Normal method, took exactly the same time—$17\frac{1}{2}$ minutes—as the Part I paper. In practice it is found that the speed of working is always increased with the second attempt. We may, therefore, regard this pupil as having taken a longer time with the Normal than with the Suggested method. It will also be noticed that, while in Part I paper with the Suggested method the pupil adopted a consistently correct procedure, the Part II paper showed a reversion to the confused method previously mentioned on p. 257. This is indicated by printing the figures in italics. It is interesting to note that this pupil (aged $13\frac{1}{2}$ years) is one of the brightest of the top class in the Senior School.

Final Tests in Long Multiplication of Money.

In order to examine as fully as possible the reliability of the preliminary tests summarised on p. 263 the following steps were taken :

(*a*) All the classes previously tested were divided into two sections as nearly as possible equal to each other in arithmetic ability.

(*b*) Both sections were given a fifteen minutes' revision lesson on 15.11.38 and a further one on 17.11.38, the one section with the Suggested method and the other with the Normal method.

Each revision lesson consisted of

(i) a model sum worked on the blackboard by teacher and pupils. This was followed by

(ii) two examples which the pupils had to try to work out for themselves. Whilst this was going on the teacher supervised, marked, and gave assistance where necessary.

The same examples for the model and practice were used with both sections.

For the purposes of revision and during the subsequent tests the classes were arranged as follows :

Normal Method.		Suggested Method.	
½ of class 3*b* ½ of class 3*a*	with class 3*b* teacher	½ of class 3*a* ½ of class 3*b*	with class 3*a* teacher
½ of class 2 ½ of class 1*a*	with class 2 teacher	½ of class 2 ½ of class 1*a*	with class 1*a* teacher

(*c*) On 18.11.38 the following tests were given to all the pupils to work according to the method in which they had been coached. The *A* questions were worked by the pupils sitting on the left-hand side of the desk and the *B* questions by those sitting on the right-hand side.

A. 1. £56 8s. 3d. × 37.
2. £64 7s. 9½d. × 53.
3. £38 15s. 6d. × 169.
4. £79 16s. 11¾d. × 183.
5. £64 18s. 7¼d. × 247.

B. 1. £65 3s. 8d. × 37.
2. £46 9s. 7½d. × 53.
3. £35 18s. 9d. × 169.
4. £69 17s. 11¼d. × 183.
5. £86 14s. 6¾d. × 247.

The Results of the Final Tests.

These are recorded below. They show the Suggested method to be 9·5 per cent. more accurate and 6·5 per cent. quicker than the Normal method. Thus they are in complete agreement with the results of the Preliminary Tests.

Method	Normal				Total	Suggested				Total
Class	3*a*	3*b*	2	1*a*		3*a*	3*b*	2	1*a*	
No. of papers	10	18	18	20	66	10	18	18	20	66
No. of examples	50	90	90	100	330	50	90	90	100	330
No. correct	36	32	37	53	158	41	46	39	47	173
Average correct	3·6	1·8	2·1	2·7	2·4	4·1	2·6	2·2	2·4	2·6
Total time	187	472	507·5	631	1797·5	179·5	407	495	600·5	1682
Average time per paper	18·7	26·2	28·2	31·6	27·2	18·0	22·6	27·5	30·0	25·5

The Suggested method shows improvements over the Normal method of :

(i) 9·5 per cent. in accuracy, and (ii) 6·5 per cent. in speed.

Preliminary Test, Part I.

(Pupil's Script.)

1. £17 5s. 8d. × 63.

		63	
£17	5	8	
1088	17	0	✓
17	42	12) 504	
441	315	42	
63	20) 357		
1088	17·17		

2. £36 16s. $5\frac{1}{2}$d. × 87.

		87	
£36	16	$5\frac{1}{2}$	
3203	11	$10\frac{1}{2}$	✓
71	39	43	
522	522	435	
261	87	12) 478	
3203	20) 1431	39	
	71·11		

3. £24 18s. $11\frac{1}{4}$d. × 159.

		159		
£24	18	$11\frac{1}{4}$		
4538	19	$0\frac{3}{4}$	×	f.
722	149	39		4) 159
636	1590	159		39
318	1272	159		
	20) 14459	12) 1788		
	722·19	149		

[Error due to incorrect positioning of figures in shillings column.]

4. £73 14s. $9\frac{3}{4}$d. × 256.

		256		
£73	14	$9\frac{3}{4}$		
18877	12	0	✓	
189	208	192		4) 768
768	2560	2304		192
1792	1024	12) 2496		
18877	20) 3792	208		
	189·12			

5. £156 13s. $8\frac{3}{4}$d. × 347.

		347		
£156	13	$8\frac{3}{4}$		
54370	4	$0\frac{1}{4}$	√	
238	253	260		4) 1041
2082	3470	2776		260
1735	1041	12) 3036		
347	20) 4764	253		
54370	238·4			

Preliminary Test, Part II.

(Pupil's script : figures in italics indicate that the money items have been used as multiplicands.)

1. £16 14s. 9d. × 57.

£	s.	d.	
16	14	9	
		57	
954	0	9	√
42	42	12) 513	
112	*98*	42	
80	*70*		
954	20) 840		
	42		

2. £42 17s. $3\frac{1}{2}$d. × 93.

£	s.	d.		
42	17	$3\frac{1}{2}$		
		93		
3986	8	$1\frac{1}{2}$	√	
80	27	46		2) 93
126	*51*	279		46
378	*153*	12) 325		
3986	20) 1608	27		
	80·8			

3. £28 12s. $10\frac{1}{4}$d. × 176.

£	s.	d.		
28	12	$10\frac{1}{4}$		
		176		
5041	2	4	√	
113	150	44		f.
168	*72*	1760		4) 176
196	*84*	12) 1804		44
28	*12*	150		
5041	20) 2262			
	113			

4. £56 13s. $8\frac{3}{4}$d. × 247.

£	s.	d.		
56	13	$8\frac{3}{4}$		
		247		
13901	11	$1\frac{1}{4}$	×	
169	180	185		4) 741
392	2470	1976		185
224	741	12) 2161		
112	20) 3391	180		
13901	169			

5. £173 14s. $9\frac{3}{4}$d. × 358.

£	s.	d.		
173	14	$9\frac{3}{4}$		
		258		
46099	2	$10\frac{1}{2}$	×	
265	290	268		4) 1074
1384	3580	3222		268
865	1432	12) 3490		
358	20) 5302	290		
46099	265			

PART I

Method	Normal Wholesale Method (see p. 256)				Total	The Suggested Method (see p. 257)				Total
Date of Test	2.11.38	9.11.38	9.11.38	7.11.38		2.11.38	9.11.38	9.11.38	7.11.38	
Class	3A	3B	2	1A		3A	3B	2	1A	
No. of papers	10	14	17	20	61	10	16	18	19	63
No. of examples	50	70	85	100	305	50	80	90	95	315
No. correct	31	19·5	29	25	104·5	33	25	23·5	27·5	109·0
Average correct	3·1	1·4	1·7	1·3	1·71	3·3	1·6	1·3	1·4	1·73
Total time (mins.)	215·5	—	565·5	816·5	1597·5	218	—	578	723	1519
Average time per paper (mins.)	21·6	—	33·3	40·8	34·0	21·8	—	32·1	38·1	32·3

PART II

Method	Normal Wholesale Method				Total	The Suggested Method				Total
Date of Test	2.11.38	11.11.38	11.11.38	8.11.38		2.11.38	11.11.38	11.11.38	8.11.38	
Class	3A	3B	2	1A		3A	3B	2	1A	
No. of papers	10	16	18	19	63	10	14	17	20	61
No. of examples	50	80	90	95	315	50	70	85	100	305
No. correct	26	27	30	29·5	112·5	29	24	31	33	117
Average correct	2·6	1·7	1·7	1·6	1·79	2·9	1·7	1·8	1·7	1·92
Total time (mins.)	191	411	515·5	644	1761·5	197·5	368·5	503	670·5	1739·5
Average time per paper (mins.)	19·1	25·7	28·6	33·9	28·0	19·8	26·3	29·6	33·5	28·5

COMBINED RESULTS OF PARTS I AND II

	Normal Wholesale Method				Total	The Suggested Method				Total
Class	3A	3B	2	1A		3A	3B	2	1A	
No. of papers	20	30	35	39	124	20	30	35	39	124
No. of examples	100	150	175	195	620	100	150	175	195	620
No. correct	57	46·5	59	54·5	217	62	49	54·5	60·5	226
Average correct	2·9	1·55	1·69	1·40	1·750	3·1	1·63	1·56	1·55	1·823
Total time (mins.)	406·5	16 papers 411	1081	1460·5	108 papers 3359	415·5	14 papers 368·5	1081	1393·5	110 papers 3258·5
Average time per paper (mins.)	20·3	16 papers 25·7	30·9	37·4	108 papers 31·1	20·8	14 papers 26·3	30·9	35·7	110 papers 29·6

The Suggested Method is 4 per cent. more accurate, and 3 per cent. quicker than the Normal.

N.B.—Owing to an unfortunate misunderstanding the teacher in charge of Part I Tests for Class 3B failed to record the times taken by each of the pupils to complete the test.

The Practice Method of Long Multiplication of Money.

The Practice method should be taught to all pupils. Its introduction, however, should be postponed until the pupil reaches the latter stages of the Junior School or the first year of a Senior School.

Example 1. What is the cost of 57 tons at £2 13s. 9d. per ton?

		£	s.	d.
Cost at £1 per ton	=	57	0	0
Cost at £2 per ton (twice cost at £1)	=	114	0	0
Cost at 10s. per ton ($\frac{1}{2}$ cost at £1)	=	28	10	0
Cost at 2s. 6d. per ton ($\frac{1}{4}$ cost at 10s.)	=	7	2	6
Cost at 1s. 3d. per ton ($\frac{1}{2}$ cost at 2s. 6d.)	=	3	11	3
Cost at £2 13s. 9d. per ton	=	153	3	9

Pupils can be led to appreciate that questions such as—FIND THE COST OF 317 ARTICLES AT £1 16S. 9D. EACH—will be more easily and conveniently worked out by the practice than the Wholesale method.

The only examples for which the practice method should be used are those in which the money factor contains simple sub-multiples and/or simple sub-multiples of sub-multiples of £1. Thus practice should only be regarded as an alternative and shorter method for long multiplication of money.

The use of the practice method becomes almost a necessity when dealing with compound examples such as—FIND THE COST OF 13 TON 15 CWT. 2 QR. AT £1 6S. 8D. PER TON? This work on compound practice, as it is called, is usually best postponed until the second year of the Senior School. Often it is dealt with at an earlier stage than this but the inclusion of such items in the earlier stages means that there cannot be sufficient and adequate practice in the revision and extension of the work already taken. The usual form of setting out compound practice is as follows :

Example 2. Find the cost of 13 ton 15 cwt. 2 qr. at £1 6s. 8d. per ton.

		£	s.	d.
Cost of 1 ton	=	1	6	8
				13
Cost of 13 tons	=	17	6	8
Cost of 10 cwt.	=		13	4
Cost of 5 cwt.	=		6	8
Cost of 2 qr.	=			8
Cost of 13 ton 15 cwt. 2 qr.	=	18	7	4

As a result of the work recorded in the previous pages of this section I have introduced into my school the following form :

Example 3.

			13
Cost of 1 ton	=£1	6	8
Cost of 13 tons	=17	6	8
Cost of 10 cwt.	=	13	4
Cost of 5 cwt.	=	6	8
Cost of 2 qr.	=		8
Cost of 13 ton 15 cwt. 2 qr.	=18	7	4

This form is a more convenient and better balanced one than its contemporary in Example 2 on the previous page. Further, when commencing compound practice in that Example 2 the first line is generally set out as

Cost of 1 ton =£1 6s. 8d.

The next line is

Cost of 13 tons =............

Now if, in Example 2 (and this often happens), the pupil has forgotten to leave the necessary space for the multiplier—13—then the appearance of his arrangement will be far from orderly. All this can be avoided by placing the number factor above the money factor as in the latter example (Example 3).

III. (*a*). FORMAL ARRANGEMENT IN LONG MULTIPLICATION OF MONEY.

Formal Arrangement in Arithmetic.

Introduction.

It is generally accepted among teachers that there are certain forms of arrangement in arithmetic with which pupils are more comfortable and more sure. Addition, for example, is usually considered to be more rapid and more accurate when the addenda are placed beneath each other as in vertical tots than when they are placed side by side as in across tots.

When considering the advantages and disadvantages of the different methods used in long multiplication of money I decided to commence with an investigation into the various forms of setting out short multiplication of money. The surprising results obtained led me to extend the work to other items of the arithmetic syllabus. Each item of the whole investigation is recorded in the chronological order with which it was dealt.

(i) Formal Arrangement in Short Multiplication of Money.

(*a*) Each class of pupils to be tested was divided into three sections as nearly as possible equal to one another in arithmetical ability. The pupils were then arranged into *a*, *b*, and *c* columns. Any odd children over were arranged as conveniently as possible :

e.g.	*a*	*b*	*c*	*a*	*b*	*c*		
	a	*b*	*c*	*a*	*b*	*c*		
	a	*b*	*c*	*a*	*b*	*c*		*c*
	a	*b*	*c*	*a*	*b*	*c*	*b*	
	a	*b*	*c*	*a*	*b*	*c*	*a*	

(*b*) Examples of the three forms to be tested ((i), (ii), and (iii) below) were written on the blackboard and the teacher indicated that in each form the number and not the money was to be used as the multiplier. This indication was, of course, particularly essential in the case of form (iii).

(i)				(ii)		(iii)			
	£3	7	$4\frac{1}{2}$	£7 4 $6\frac{3}{4} \times 5 =$ ________					7
			6				£6	3	$7\frac{1}{2}$

(*c*) Next, the pupils were given test papers containing a number of examples with the spaces, in which they had to write the answers, marked. Copies of these test papers are given on pages 266-7. The pupils in section *a* were given Test *a* papers, those in section *b* Test *b* papers, and Test *c* papers were given to those in section *c*. The instructions at the top of the papers told the pupils to fill in as many answers as possible in ten minutes. The "Go!" and "Stop!" signals were given by the teachers.

(*d*) When the first lot of test papers had been collected, those pupils who had formerly worked Test *a* paper were now given Test *b* paper, whilst those who had formerly worked Test *b* and *c* papers were given Test *c* and *a* papers respectively. This second set of papers was worked through in exactly the same manner as the first.

(*e*) Finally the pupils worked through the third set of papers containing the examples with which they had not already dealt.

Assessing the Tests.

Each paper was examined twice. On the first examination a mark was given for every completed answer. Incomplete answers received one-third of a mark if the pence column had been finished and a further third of a mark if the shillings' column had been fully dealt with. On the second examination every correct answer received one mark. Incomplete answers received a third of a mark for a correct pence column and two-thirds if both shillings and pence columns were correct.

Test *a* Paper.

Here are 20 * short multiplication of money sums. The first is worked out for you. Study it for one minute and then see how many of the remainder you can work out in TEN MINUTES.

* Tests *a*, *b*, *c* each contained 20 sums, but only 3 of each set need be reproduced here.

All you have to do is to fill in the answers.

	£	s.	d.			£	s.	d.			£	s.	d.
1.	6	7	$4\frac{1}{2}$		2.	3	9	$10\frac{1}{4}$		3.	5	8	$9\frac{3}{4}$
			5					4					3
	31	16	$10\frac{1}{2}$										

Test *b* Paper.

Here are 20 short multiplication of money sums. The first is worked out for you. Study it for one minute and then see how many of the remainder you can work out in TEN MINUTES.

All you have to do is to fill in the answers.

1. £3 9s. $10\frac{1}{4}$d. × 5 = £17 9s. $3\frac{1}{4}$d.
2. £6 7s. $4\frac{1}{2}$d. × 4 = ________
3. £5 8s. $9\frac{3}{4}$d. × 6 = ________

Test *c* Paper.

Here are 20 short multiplication of money sums. The first is worked out for you. Study it for one minute and then see how many of the remainder you can work out in TEN MINUTES.

All you have to do is to fill in the answers.

1.			6		2.			3		3.			5
	£3	7	$4\frac{1}{2}$			£6	9	$10\frac{1}{4}$			£7	4	$7\frac{3}{4}$
	20	4	3										

Results of Tests in Formal Arrangement in Short Multiplication of Money.

Class	No. of papers	Speed			Accuracy		
		No. of questions answered			No. of correct answers		
		Test *a*	Test *b*	Test *c*	Test *a*	Test *b*	Test *c*
3*a*	20 of each	$316\frac{2}{3}$	291	$332\frac{2}{3}$	$246\frac{2}{3}$	$242\frac{1}{3}$	270
3*b*	39 ,,	399	378	$433\frac{1}{3}$	$258\frac{2}{3}$	$243\frac{1}{3}$	293
2	41 ,,	$366\frac{2}{3}$	$325\frac{1}{3}$	$365\frac{2}{3}$	$227\frac{1}{3}$	184	228
1*a*	39 ,,	$305\frac{2}{3}$	293	319	206	$191\frac{1}{3}$	$199\frac{2}{3}$
1*b*	33 ,,	143	133	162	33	$35\frac{2}{3}$	$50\frac{1}{3}$
H.H.J. 4	48 ,,	396	$357\frac{2}{3}$	$431\frac{1}{3}$	$243\frac{2}{3}$	$226\frac{1}{3}$	278
N.C.J. 4	14 ,,	$62\frac{2}{3}$	47	63	$26\frac{1}{3}$	28	$36\frac{2}{3}$
Totals	234 of each	$1989\frac{2}{3}$	1825	2107	$1241\frac{2}{3}$	1151	$1355\frac{2}{3}$

c form was (i) 5·9 per cent. more accurate and 9·2 per cent. quicker than the a form, and

(ii) 15·5 per cent. more accurate and 17·8 per cent. quicker than the b form.

a form was 9·0 per cent. more accurate and 7·8 per cent. quicker than the b form.

Observations on the Tests.

Before giving the tests I was of the opinion that form (i) p. 266 would give the best result both as regards speed and accuracy. I was, therefore, very surprised to find that these honours went convincingly to form (iii), p. 266.

The results showed a high degree of correlation between speed and accuracy.

In the matter of speed and accuracy form (ii) provided the poorest results. This is probably the largest of the contributory factors which cause the Ten-ten method of long multiplication of money (see Part II, p. 255) to be slower and more inaccurate than the Wholesale method.

Shortly after the tests had been concluded I went round to all the classes and questioned the children on how they had carried out their multiplications. I found that in Test a and Test b papers all the children had used the number as the multiplier, but in the case of Test c papers about one-third of them had used the money items as multipliers whenever possible. This forced me to the conclusion that, had all pupils used the number as the multiplier, form (iii) on Test c paper would have shown an even greater superiority over the other two forms both in speed and accuracy.

It seems that, in the early stages, work on short multiplication of money would be made easier if the general arrangement adopted were that of placing the multiplier above the sum of money.

This effect of placing the multiplier above the multiplicand in short multiplication of money is not altogether in accord with the improved results obtained with the Suggested method of long multiplication of money (see the work on long multiplication of money, Part II), since in the one case the number is used as the multiplier and in the other, as the multiplicand.

We have evidence that A and B forms below give better results than C and D respectively :

A. 6 £7 16 $4\frac{1}{2}$	*B.* 247 £7 16 $4\frac{1}{2}$
C. £7 16 $4\frac{1}{2}$ 6	*D.* £7 16 $4\frac{1}{2}$ 247

although the multiplications are carried out in different directions

(see arrow-heads in the examples). It appears, therefore, that the better results obtained by the *A* and *B* forms must be due to arrangement and not method. There are two possible explanations of this :

(i) The factors of these multiplications are heterogeneous (number and money) quantities. In the *A* and *B* forms the like quantities—product and money factor—are kept together, whilst in *C* and *D* they are separated by the number factor. Further, this separation is likely to rob the pupil of the assistance of alignment (see paragraph at the foot of page 257, line 9 up, of the work on long multiplication of money). The fact that the pupil can put like quantities immediately under each other seems to simplify the mental operations in which he is involved.

(ii) In the *A* and *B* forms both factors are conspicuous. In the *C* and *D* forms the number factor is somewhat obscured by the lengthy money factor above and the answer lines beneath.

Final Notes on Formal Arrangement in Short Multiplication of Money.

It must not be gathered, from the results of my tests on Formal Arrangement, that I suggest that the more successful arrangements should always be adhered to. I consider it excellent practice for pupils to add and subtract numbers which are not necessarily arranged under one another, and to carry out short multiplication without even setting the sums out in the usual formal manner. When, however, more complicated rules of arithmetic are being learned the initial burden is lightened if the best possible formal arrangement is adopted.

III. (*b*). FORMAL ARRANGEMENT IN REDUCING SHILLINGS TO POUNDS.

There are two forms of setting out questions involving the reduction of shillings to £'s. They are :

(i) $2\cancel{0}\,)\,345\cancel{1}$ (ii) $20\,)\,3451$

£172 11s. and £172 11s.

In example (i), after crossing off the nought of the 20 and the last figure of the dividend, division is by 2 ; whilst in example (ii) division is by 20. Consequently, in (ii), the first figure of the quotient is one place further to the right than it is in (i). The divisor, 20, is often omitted in both examples.

All examples of long multiplication of money given in the Mathematical Association's *Report on the Teaching of Arithmetic in Schools* * (Bell, 1937) use the form in example (ii) above. Dr. Ballard in his *Teaching the Essentials of Arithmetic* † (University of London Press, 1928) gives both forms.

* pp. 20-21. † pp. 132-133.

Tests in Formal Arrangement when reducing Shillings to £'s.

In order to discover which of the two forms set out above gives the better result I carried out tests in the following manner :

1. Prior to the tests which were given on 2.12.38 two lessons on the 20 times multiplication and division tables from 1 to 9 were given on 29.11.38 and 1.12.38 respectively. At the end of these lessons the majority of the pupils had a 100 per cent. knowledge of the tables. Copies of the instructions for these lessons are given on pp. 273 and 274.
2. For the tests every class was divided into two sections as nearly as possible equal to each other in arithmetical ability. One section sat in the left-hand desk positions and the other in the right-hand positions.
3. Every class was shown both forms ((i) and (ii) above) of carrying out reduction of shillings to £'s.
4. Next, the pupils on the left-hand side of the desks (see 2 above) were given Test *A* papers to work and those on the right-hand side Test *B* papers. Test *A* paper contained examples to be worked according to examples (i) on the previous page, and Test *B* papers examples to be worked according to example (ii). As will be seen from the instructions at the head of the test papers (see p. 273) the pupils were allowed four minutes to complete as many of the examples as possible. A further instruction issued to the teachers was that if any pupil completed either test paper before the four minutes had elapsed then the whole class was to be stopped at that point. It was only necessary to bring this instruction into force with the top class.
5. After the first set of test papers has been collected, pupils sitting on the left-hand side of the desks were given Test *B* papers, and those on the right-hand side Test *A* papers to work under the conditions set out in 4 above.

Assessing the Tests.

As in previous tests on formal arrangement each paper was examined twice. On the first examination one mark was given for every complete answer, and on the second examination each correct answer received one mark.

Incomplete answers were ignored.

Observations on the Tests.

The results of the tests showed that when reducing shillings to £'s the method of crossing off the nought of the divisor and the last figure of the dividend gives :

(i) a far better result both as regards speed and accuracy. (It was 58 per cent. quicker and 79 per cent. more accurate than the method shown in example (ii), page 269.)

(ii) a higher degree of accuracy. The figures in support of this are :

FIRST SET OF TESTS.

(*a*) 79 per cent. of Test *A* paper examples answered were correct.
(*b*) 77 per cent. of Test *B* paper examples answered were correct.

SECOND SET OF TESTS.

(*a*) 87 per cent. of Test *A* paper examples answered were correct.
(*b*) 71 per cent. of Test *B* paper examples answered were correct.

COMBINED RESULTS.

(*a*) 84 per cent. of Test *A* paper examples answered were correct.
(*b*) 74 per cent. of Test *B* paper examples answered were correct.

THE FOOL-PROOF FORM.

One of the greatest sources of error in reducing shillings to £'s is the remainder. This liability to error can be greatly reduced by using the form in example (i) p. 269 (already shown to give the better results), and carrying out the operation in the following stages :

(*a*) $2\cancel{0}\ )\ \underline{345\cancel{1}}$
$\underline{\qquad 1s.}$

Cross off the nought and the last figure of the shillings. Write the shillings figure crossed off down in the space for the remainder.

(*b*) $2\cancel{0}\ )\ \underline{345\cancel{1}}$
$\underline{£172\ 11s.}$

Now divide by 2 and write the number over (this can never be more than 1) to the left of the figure already in the shillings remainder.

RESULTS OF THE FIRST SET OF TESTS ON FORMAL ARRANGEMENT IN THE REDUCTION OF SHILLINGS TO £'S

FORM	NO. OF PAPERS	SPEED		ACCURACY	
		No. of questions answered		No. of correct answers	
		Test *A*	Test *B*	Test *A*	Test *B*
3*a*	11*A* 10*B*	200	120	190	102
3*b*	18*A* 18*B*	256	182	190	141
2	21*A* 21*B*	257	139	208	111
1*a*	19*A* 19*B*	252	130	217	107
1*b*	17*A* 16*B*	107	55	45	26
Totals	86*A* 84*B*	1072	626	850	487

RESULTS OF THE SECOND SET OF TESTS ON FORMAL ARRANGEMENT IN THE REDUCTION OF SHILLINGS TO £'S

Form	No. of papers	Speed		Accuracy	
		No. of questions answered		No. of correct answers	
		Test *A*	Test *B*	Test *A*	Test *B*
3*a*	10*A* 11*B*	195	167	191	155
3*b*	18*A* 18*B*	356	220	315	154
2	21*A* 21*B*	297	220	239	156
1*a*	19*A* 19*B*	246	136	228	100
1*b*	16*A* 17*B*	121	74	87	15
Totals	84*A* 86*B*	1215	817	1060	580

COMBINED RESULTS OF THE FIRST AND SECOND SETS OF TESTS

Form	No. of papers	Speed		Accuracy	
		No. of questions answered		No. of correct answers	
		Test *A*	Test *B*	Test *A*	Test *B*
3*a*	21*A* 21*B*	395	287	381	257
3*b*	36*A* 36*B*	612	402	505	295
2	42*A* 42*B*	554	359	447	367
1*a*	38*A* 38*B*	498	266	445	207
1*b*	33*A* 33*B*	228	129	132	41
Totals	170*A* 170*B*	2287	1443	1910	1067

Formal Arrangement in reducing Shillings to £'s.

Test *A*.*

Your teacher has shown you how to work out the questions on this paper. The first is worked out for you. Use it as a guide and see how many of the remainder you can work out in FOUR MINUTES. All you have to do is to fill in the answers.

1. 2$\cancel{0}$) 345$\cancel{1}$	2. 20) 1467	3. 20) 1892	4. 20) 1653
£172 11s.	______	______	______

Formal Arrangement in reducing Shillings to £'s.

Test *B*.

Your teacher has shown you how to work out the questions on this paper. The first is worked out for you. Use it as a guide and see how many of the remainder you can work out in FOUR MINUTES. All you have to do is to fill in the answers.

1. 20) 3451	2. 20) 1764	3. 20) 1298	4. 20) 1563
£172 11s.	______	______	______

Lessons in preparation for Tests in Formal Arrangement in connection with reduction of Shillings to Pounds Sterling.

N.B.—The tables and items *A* and *B* in the lessons below should be written on the blackboards before the lessons commence.

Lesson 1.

$20 \times 1 = 20$
$20 \times 2 = 40$
$20 \times 3 = 60$
$20 \times 4 = 80$
$20 \times 5 = 100$
$20 \times 6 = 120$
$20 \times 7 = 140$
$20 \times 8 = 160$
$20 \times 9 = 180$

1. Allow the pupils from 3-5 minutes to learn the multiplication table shown here. They should do this by repeating to themselves individually each item of the table.

2. At the end of the period allotted for this memorisation, spend two or three minutes asking questions on the table around the class, allowing the pupils to make reference to the table if necessary.

3. Allow the pupils $2\frac{1}{2}$ minutes to work through Test *A*, and permit them to make reference to the table if they wish. Spend 2 minutes in marking the test, each pupil marking his own work.

A. 20×4 ;	20×6 ;	20×1 ;	20×9 ;	20×7 ;	20×2 ;
20×8 ;	20×3 ;	20×5 ;	20×1 ;	20×4 ;	20×7 ;
20×6 ;	20×9 ;	20×3 ;	20×5 ;	20×2 ;	20×8.

4. Allow the pupils $2\frac{1}{2}$ minutes to work through Test *B*, but in this case do not allow them to make reference to the table. Spend 2 minutes in marking the test, each pupil marking the work of the pupil sitting beside him.

* Tests *A* and *B* each contained 24 sums, of which only the first four of each set are reproduced here.

B.	20×8 ;	20×2 ;	20×4 ;	20×7 ;	20×3 ;	20×1 ;
	20×9 ;	20×6 ;	20×8 ;	20×9 ;	20×1 ;	20×6 ;
	20×5 ;	20×2 ;	20×4 ;	20×7 ;	20×5 ;	20×3.

LESSON 2.

$20 \div 20 = 1$
$40 \div 20 = 2$
$60 \div 20 = 3$
$80 \div 20 = 4$
$100 \div 20 = 5$
$120 \div 20 = 6$
$140 \div 20 = 7$
$160 \div 20 = 8$
$180 \div 20 = 9$

1. Allow the pupils from 3-5 minutes to learn the division table shown here. They should do this by repeating to themselves each item of the table.
2. At the end of the period allotted for this memorisation, spend 2 or 3 minutes asking questions round the class. Allow the pupils to make reference to the table if necessary.

3. Allow the pupils $2\frac{1}{2}$ minutes to work through Test *A*, and permit them to make reference to the table if they wish. Spend 2 minutes in marking the test, each pupil marking his own work.

A.	$80 \div 20$;	$120 \div 20$;	$20 \div 20$;	$180 \div 20$;	$140 \div 20$;
	$40 \div 20$;	$160 \div 20$;	$60 \div 20$;	$100 \div 20$;	$20 \div 20$;
	$80 \div 20$;	$140 \div 20$;	$120 \div 20$;	$180 \div 20$;	$60 \div 20$;
	$100 \div 20$;	$40 \div 20$;	$160 \div 20$.		

4. Give the pupils $2\frac{1}{2}$ minutes to work through Test *B*, but do not allow them to make reference to the table. Spend 2 minutes in marking the test, each pupil marking the work of the pupil sitting beside him.

B.	$160 \div 20$;	$40 \div 20$;	$80 \div 20$;	$140 \div 20$;	$60 \div 20$;
	$20 \div 20$;	$180 \div 20$;	$120 \div 20$;	$160 \div 20$;	$180 \div 20$;
	$20 \div 20$;	$120 \div 20$;	$100 \div 20$;	$40 \div 20$;	$80 \div 20$;
	$140 \div 20$;	$100 \div 20$;	$60 \div 20$.		

H. W.

1328. Next, we may ask where the climax, if there is to be one at all, should occur. There is a beautiful theory about this for the high light in a picture, or the "jewel that on the stretched forefinger of all time sparkles for ever" in a poem, or the spire of a church, or the turning point of a tune : it is called the Golden Section, and it says that the small part, *a*, is to the greater, *b*, as *b* is to the whole. Those wonderful fellows, the mathematicians, state this neatly as

$$\frac{a}{b} = \frac{b}{a+b}$$

and show *b* to be approximately ·618 of the whole. L. Sabaneev, who has written a deal of good sense about music, found this to be correct for much of Chopin. The trouble about it is that painter, poet and musician all maintain that it is only true with reserves, if at all ; and their reason may well be that, as has been already said, climax is not a moment but a process. But that it comes about two-thirds of the way through is only natural, because it must have time to build itself up, and the excitement must die down again for us to understand it as a climax.—*Observer*, February 26, 1939.

MATHEMATICAL NOTES.

1472. *A Note on implicit functions.*

Given a set of n equations in an independent variable x and n dependent variables $y_1, y_2, \dots, y_n$ satisfied by simultaneous values $a, b_1, b_2, \dots, b_n$, we are prone to think that "in practice" the existence of $y_1 - b_1, y_2 - b_2, \dots, y_n - b_n$ as functions of $x - a$ can be taken for granted, and that the calculation of as many leading coefficients as are wanted is a mechanical process; only perverted ingenuity constructs cases of failure. Consider, however, the following example, which is a straightforward problem in applied mathematics.

A weight W is hung from two pegs on the same level at distance $2a$ apart by two uniform flexible strings each of length l and weight wl; if w is not negligible but is small compared with W, find approximations to the parameter c of the catenaries along which the strings lie and to the angles ψ, χ which these catenaries make with the horizontal at the point of attachment of W and at the pegs.

If x is the horizontal distance between the point of attachment of W and the vertex of a catenary, a set of exact equations is

$$2wc \tan\psi = W,\ l = c(\tan\chi - \tan\psi),\ \operatorname{ch} x/c = \sec\psi,\ \operatorname{ch}(x+a)/c = \sec\chi.$$

Writing $$2w/W = t,\ 1/c = y,\ x/c = u,$$

in order to avoid values which are essentially large, we have

$$y = t\tan\psi,\ ly = \tan\chi - \tan\psi,\ \operatorname{ch} u = \sec\psi,\ \operatorname{ch}(u+ay) = \sec\chi,$$

and we can say that the problem is to determine ψ, χ, u from the set of equations:

$$lt = \cot\psi\tan\chi - 1,\ \operatorname{ch} u = \sec\psi,\ \operatorname{ch}(u + at\tan\psi) = \sec\chi,$$

the subsequent determination of y being trivial.

When $t = 0$, the three equations reduce to two, and the set is satisfied by
$$\psi = \chi = \beta,\ u = f,$$

provided only that f is given by $\operatorname{ch} f = \sec\beta$; it appears that β is an arbitrary acute angle. In relation to the problem proposed, this result is manifestly absurd: in the limiting case of light strings, ψ and χ have the same value, but this value is not arbitrary, for necessarily the angle has cosine a/l.

Let us attempt to develop a solution of the set of equations from the proposed initial values. We have

$$\cot\psi\sec^2\chi\frac{d\chi}{dt} - \operatorname{cosec}^2\psi\tan\chi\frac{d\psi}{dt} = l,$$

$$\sec\psi\tan\psi\frac{d\psi}{dt} - \operatorname{sh} u\frac{du}{dt} = 0,$$

$$\sec\chi\tan\chi\frac{d\chi}{dt} - \left(at\sec^2\psi\frac{d\psi}{dt} + \frac{du}{dt} + a\tan\psi\right)\operatorname{sh}(u + at\tan\psi) = 0,$$

and therefore for initial values of first derivatives, replacing $\operatorname{ch} f$ and $\operatorname{sh} f$ by their values $\sec\beta$, $\tan\beta$,

$$\frac{d\psi}{dt}-\frac{d\chi}{dt}=-l\sin\beta\cos\beta,$$

$$\frac{d\psi}{dt}=\cos\beta\frac{du}{dt},\quad \frac{d\chi}{dt}=\cos\beta\frac{du}{dt}+a\sin\beta.$$

These equations are not independent, and they are inconsistent unless $l\cos\beta=a$, the relation predicted from the statical problem. When this relation is satisfied, we have only two equations for the three derivatives and the initial values of the derivatives are not immediately determinate, but a third equation between the initial values of first derivatives emerges at the next stage as a condition of consistency between three related equations involving initial values of second derivatives. Actually this third equation reduces readily to

$$\cos^2\beta\left(\frac{d\chi}{dt}+\frac{d\psi}{dt}\right)=2\cos\beta\frac{du}{dt}+a\sin\beta,$$

implying the initial values

$$\frac{du}{dt}=-\tfrac{1}{2}a\tan\beta,\quad \frac{d\chi}{dt}=-\frac{d\psi}{dt}=\tfrac{1}{2}a\sin\beta,$$

whence

$$\cos\psi=\frac{a}{l}+\frac{a(l^2-a^2)}{l^3}\cdot\frac{w}{W}+O\left(\frac{w}{W}\right)^2,\quad \cos\chi=\frac{a}{l}-\frac{a(l^2-a^2)}{l^3}\cdot\frac{w}{W}+O\left(\frac{w}{W}\right)^2,$$

$$2c\sqrt{(l^2-a^2)}=\frac{W}{w}\cdot a+la+O\left(\frac{w}{W}\right).$$

The gist of this matter, as far as it depends on the determination of an implicit function, is already present in such an elementary example as the definition of y as a function of x by the equation

$$xy=\sin x,$$

which fails to determine the value of y when $x=0$. The assumption that y can be differentiated involves the assumption that y is continuous, and therefore that the value of y at $x=0$ is the limiting value 1; alternatively, differentiating the equation we have

$$x\frac{dy}{dx}+y=\cos x,$$

and if the derivative is to be finite at $x=0$, the value of y there must be 1. Here suddenly our difficulties take an acute form, for if we regard our last equation as a differential equation we are bound to accept

$$xy=\sin x+C$$

as a general integral in spite of the fact that if the integral is to exist at $x=0$, the constant can have no value but zero.

This is not the place for a systematic account of the problems that have been raised. The objects of this Note are first, to show that these problems arise naturally, and secondly, to illustrate a sort of delayed action that is characteristic of the determination of implicit functions near exceptional points. The mathematician does not read into a simple equation like $xy = \sin x$ elaborate difficulties which are not "really" there; he uses simple examples in order to cope one by one with difficulties which he is compelled to master sooner or later.

E. H. N.

1473. *The potential of a rod.*

The potential V of a uniform rod AB at a point P is $\gamma\rho\int dx/r$, where, if x is measured from the projection O of P on AB and if $OP = k$, $r^2 = k^2 + x^2$. Writing $x = k \operatorname{sh} \phi$, $r = k \operatorname{ch} \phi$, we have immediately $V = \gamma\rho\,[\phi]$, where $r + x = ke^{\phi}$ and therefore $\phi = \log\{(r+x)/k\}$.

If A is between O and B, and if $OA, OB = a, b$, we have

$$[\phi] = \log\{(r_2 + b)/(r_1 + a)\},$$

and since $r_2{}^2 - b^2 = r_1{}^2 - a^2$,

$$\frac{r_2 + b}{r_1 + a} = \frac{r_1 - a}{r_2 - b} = \frac{r_1 + r_2 + (b - a)}{r_1 + r_2 - (b - a)}.$$

If O is between A and B,

$$[\phi] = \log\{(r_2 + b)(r_1 + a)/k^2\} = \log\{(r_2 + b)/(r_1 - a)\},$$

and now
$$\frac{r_2 + b}{r_1 - a} = \frac{r_1 + a}{r_2 - b} = \frac{r_1 + r_2 + (a + b)}{r_1 + r_2 - (a + b)}.$$

Thus in either case, if l is the length of the rod,

$$V = \gamma\rho \log \frac{r_1 + r_2 + l}{r_1 + r_2 - l}.$$

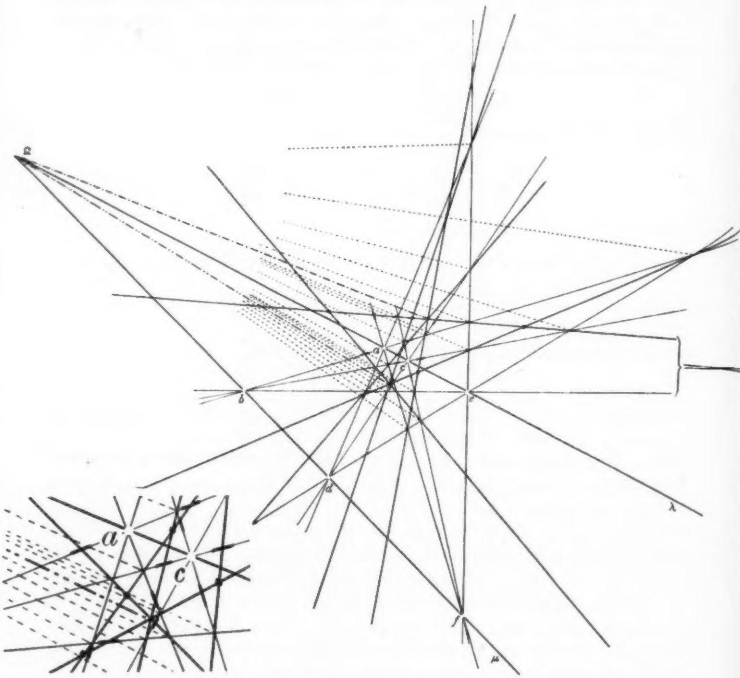
E. H. N.

1474. *Notes on Conics.* **3** : *How many Pascal lines has a sixpoint?*

How many Pascal lines are derivable, if there are no accidental coincidences, from a set of six points *abcdef* on a conic? We all know the answer, sixty, and we all know an argument by which it is often professed that the answer is supplied: a definite cyclic order of the six points determines the line uniquely, and the number of cyclic arrangements of n terms is $(n-1)!$, but since the line is not changed if the order is completely reversed, the number $5!$ must be halved. Obviously all that this argument does prove is that the required number *does not exceed 60*, for if we had noticed that the line is determined by the cyclic order but had failed to notice the possibility of reversal we should have declared, as legitimately and as confidently, that the number is 120. Salmon has the enumerative argument in another form: the number of crosspoints (ab, cd) is 45, each of these is on four lines, and each line contains three of the points. There is the same fallacy, for if there were systematic coincidences the conclusion would be false. The missing step must

belong to geometry, not to the theory of permutations, and the point of my opening question is to ask how this step is to be taken.

Sheer construction of a diagram is not an acceptable solution. A mathematical theorem can not be made more than plausible by draughtsmanship however careful, and the labour of constructing a complete figure is anyhow formidable. If we shorten the work by simplifying the conic or the sextet, we are liable to introduce coincidences by our very simplification and we can not expect to reach a total of 60 distinct lines ; for example, several of the Pascal lines of a regular sixpoint coincide in the line at infinity. To estimate coincidences and to allow for them at this stage would be of course to beg the question at issue.



Let us examine for a moment the nature of a systematic coincidence. If there is a necessary coincidence of two of the Pascal lines of a sextet, this means that *every* Pascal line belongs to two or more distinct arrangements of the vertices, whatever the nature of the conic or of the sextet. *If in any one figure there is any one Pascal line which demonstrably belongs to only one arrangement of the vertices, there are no systematic coincidences and the number of lines*

in general is truly 60. If every Pascal line belongs to two or more arrangements, the total number of arrangements, which we know to be 60, is at least twice the number of distinct Pascal lines, and we have an alternative criterion : *If in any one figure there are more than 30 distinct Pascal lines, then the number 60 is correct.*

Consider now the case of a conic formed of two lines λ, μ intersecting in a point Ω and a sextet of points of which *ace* are on λ and *bdf* are on μ, none of these points being at Ω. There are 9 *transversals*, that is, lines joining one of the points on λ to one of the points on μ, and these transversals are all distinct. We can choose in 18 ways a pair of transversals whose point of intersection is not on λ or μ, and we need the lemma that *In general the lines joining Ω to the 18 crosspoints given by these pairs of transversals are all distinct.*

Perhaps the quickest proof of this lemma is a particular metrical construction. If λ, μ are parallel lines, and if the lengths *ac*, *ce*, *ae* and *bd*, *df*, *bf* have such ratios (for example, $1:3:4$, $2:5:7$) that no two fractions with a numerator in one set and a denominator in the other are equal, the 18 lines through the crosspoints parallel to λ and μ form 9 distinct pairs of parallels harmonic with λ and μ.

Returning to the problem, we now suppose that λ and μ intersect in Ω and that the 18 lines joining Ω to the crosspoints are distinct. There are three types of hexagon, distinguished by the numbers of sides which lie along the conic-lines λ, μ :

(A) For the hexagon *acebdf*; with two sides along each conic-line, the Pascal line joins Ω to the crosspoint (*af*, *be*) ;

(B) For the hexagon *abcefd*, with one side along each conic-line, the Pascal line is the transversal *ab* itself ;

(C) For the hexagon *abcdef* the Pascal line is a genuine Pappus line through three crosspoints.

There are 18 distinct lines of type (A) and 9 distinct lines of type (B), and since no transversal passes through Ω there are no coincidences between (A) and (B). Already we have 27 distinct lines, and we need only 4 more. Since no two crosspoints are in line with Ω, no line of type (C) is also of type (A) ; since no three transversals are concurrent, no line of type (C) is also of type (B) ; lastly, there are 4 hexagons of type (C) which have one side along the transversal *ab*, namely, *abcdef*, *abcfed*, *abedcf*, *abefcd*, and since the sides opposite to *ab* in these 4 hexagons are the distinct transversals *de*, *fe*, *dc*, *fc*, the Pascal lines cut *ab* in 4 distinct points and are therefore themselves distinct. We have found 31 distinct Pascal lines in one figure, and therefore there can be no systematic coincidences and

The number of distinct Pascal lines is in general sixty.

The economy of 31 is irresistible, but in fact it is easily shown that the 6 hexagons of type (C) all have distinct lines : since each hexagon of the type includes two of the three sides *ab*, *ad*, *af*, any two of them have one of these sides in common. Thus the actual number of distinct Pascal lines in our figure is 33. The transversal *ab* belongs as a Pascal line to the four hexagons *abcefd*, *abcedf*,

abecfd, *abecdf*, and therefore in the total of 60 Pascal lines the 9 transversals account for 36 lines, and the remaining 24 lines are simple. In an alternative line of argument this is the point of attack. The crosspoint (af, be) determines uniquely the hexagon of type (A) to which it belongs, and when it is proved that a line joining Ω to a crosspoint can not be a Pascal line of a hexagon of type (B) or type (C), it follows that Pascal lines do exist which belong to only a single hexagon and that multiplicity is always accidental.

It is interesting to complete our study by recognising in our particular figure the concurrencies and collinearities associated with the names of Steiner, Salmon, and Kirkman. The Steiner triads are of two types. Each crosspoint is a Steiner point, the Pascal lines concurrent there being the two transversals and the line to Ω ; the line joining Ω to a crosspoint does not pass through any other crosspoint and occurs only once in this classification, but each transversal passes through four crosspoints and is reckoned at each of them, in agreement with its fourfold occurrence as a Pascal line. The Pappus lines form 2 concurrent triads, determining 2 more Steiner points, making with the 18 crosspoints the usual total of 20. The 15 Steiner lines are the 9 transversals and the 6 Pappus lines ; the 4 Steiner points on a transversal are the 4 crosspoints ; the 4 Steiner points on a Pappus line are the 3 crosspoints by which it is identified and its point of concurrence with the other Pappus lines of its triad.

The Kirkman triads are of three types :

(A′) The hexagon *acebdf* has two sides which are on transversals, *af*, *be* ; the two vertices which are not on these sides determine a third transversal *cd* ; the six transversals other than *af*, *be*, *cd* are the sides of a hexagon *abcfed* whose Pascal line is a Pappus line, and the Kirkman triad which corresponds to *acebdf* is this Pappus line together with the transversal *cd* duplicated ;

(B′) The hexagon *abcefd* has two sides, *ce*, *fd*, along the conic-lines, and of the two pairs of transversals joining the ends of these sides one pair includes the side *ef* ; the Kirkman triad of the hexagon consists of the two transversals *cf*, *de*, which form the second pair, together with the line joining Ω to their crosspoint (cf, de) ;

(C′) The sides of the hexagon *abcdef* are on six of the transversals ; the Kirkman triad of the hexagon consists of the lines joining Ω to the vertices of the triangle formed by the other three transversals.

The 18 distinct hexagons of type (A′) give 18 distinct Kirkman triads, and the points of concurrence of these triads are the points in which a Pappus line cuts a transversal which is not a side of the hexagon from which it is derived.

Of the four hexagons of type (B′) which have the same two sides along the conic-lines, two have the same Kirkman triad ; thus the 36 hexagons of the type give 18 distinct triads, the Kirkman points being the 18 crosspoints, each of which must be reckoned double in an enumeration.

There are 6 hexagons of type (C′); each of these involves three of the crosspoints, and the 6 together account for the 18 lines joining Ω to crosspoints; the 6 Kirkman triads are distinct, but Ω is a sextuple Kirkman point.

We can check the occurrence of the Pascal lines of the three types in the Kirkman triads. A line of type (A), that is, a line joining Ω to a crosspoint, belongs to one triad of type (B′) and to one of type (C′), and since the triad of type (B′) is double, this line is recognisably a member of 3 triads. A line of type (B), that is, a transversal, is duplicated in each of four triads of type (A′), and belongs also to two triads of type (B′); these last triads being double, the transversal is a member of 12 triads, consistently with its quadruple character as a Pascal line. A line of type (C), that is, a Pappus line, belongs to 3 triads of type (A′).

Each Pappus line appears in triplicate as a Salmon line; of the three crosspoints on the line, two function as Kirkman points and the third as a Steiner point, and one of the three intersections with a duplicated transversal is appropriated in each case: if (*ab*, *de*) and (*af*, *cd*) are Kirkman points and (*bc*, *ef*) is a Steiner point, the third Kirkman point is the intersection with *ad*. The triplicated Pappus lines account for 18 Salmon lines, and the other Salmon lines join Ω to the points of concurrence of the 2 Pappus triads; on each of these lines Ω is a triple Kirkman point: the Steiner point of concurrence of the three Pascal lines *abcdef*, *afcbed*, *adcfeb* is to be regarded as collinear with the three Kirkman points, coincident at Ω, which arise from joining Ω to the vertices of the triangles whose sides are *ab*, *de*, *cf*; *af*, *be*, *cd*; *ad*, *fe*, *cb*.

The 15 Salmon points are the 9 points in which a member of one Pappus triad cuts a member of the other, and the 6 points in which a member of one Pappus triad cuts the line joining Ω to the vertex of the other triad. For purposes of enumeration only two of the three Salmon lines coincident in a Pappus line are operative at a point of the first kind; the two intersecting Pappus lines are the 4 Salmon lines through the point, and each Salmon line counts only two of the three points of this kind through which it passes, its third Salmon point being its point of intersection with the line joining Ω to the alien vertex. The 4 Salmon lines through a point of the second kind are the triple Pappus line and the line to Ω.

One good lesson is to be learnt from this complete enumeration. In general there are 60 distinct Pascal lines and 60 distinct Kirkman points and the correlation between them is one-one. In the degenerate figure there are coincidences in both groups, but *coincidences in one group do not correspond to coincidences in the other.* The reader who can honestly say that he would never have expected them to do so has a genuine instinct for the fundamental principles of the theory of groups.

In the figure, the two conic lines and the six Pappus lines are heavy, the nine transversals are light; heavy long-and-short-dash lines join Ω to the two Pappus concurrences, and lines in light dashes converge towards

Ω from the eighteen crosspoints. The nine Salmon points in which a member of one Pappus triad cuts a member of the other are marked heavily; each transversal is thickened near the two Kirkman points where it cuts a Pappus line with which it is not directly associated.

In the inset lower diagram, part of the figure is reproduced on the scale on which the diagram was in fact drawn, and detail is shown which is lost in the reduction of the complete figure.

E. H. N.

1475. *The foci of conics.*

1. *The focus and directrix of the parabola*

$$(2x+3y)^2+4x+5y+6=0. \qquad \text{(i)}$$

Since the axis is parallel to $2x+3y=0$, the directrix is parallel to $3x-2y=0$; take it to be $3x-2y+c=0$, and the focus as (h, k).

The equation to the parabola is therefore

$$(3x-2y+c)^2=13(x-h)^2+13(y-k)^2,$$

or $$(2x+3y)^2-x(26h+6c)-y(26k-4c)+13h^2+13k^2-c^2=0. \qquad \text{(ii)}$$

Comparing (i) and (ii),

$$26h+6c=-4,$$
$$26k-4c=-5,$$
$$13(h^2+k^2)-c^2=6.$$

Substituting for h and k gives us

$$(6c+4)^2+(4c-5)^2-52c^2=312$$

and $$8c=271.$$

Hence $$h=-829/104, \quad k=261/52.$$

This is the focus; and the directrix is

$$24x-16y+271=0.$$

The axis, being perpendicular to the directrix and passing through the focus, is

$$52x+78y+23=0.$$

2. *The foci of the central conic*

$$ax^2+2hxy+by^2=1. \qquad \text{(i)}$$

The foci lie on the axes, which are the bisectors of the angles between the asymptotes.

Hence their coordinates satisfy

$$(x^2-y^2)/(a-b)=xy/h. \qquad \text{(ii)}$$

Now the line $x=x_1$ will touch (i) if

$$4h^2x_1^2=4b(ax_1^2-1).$$

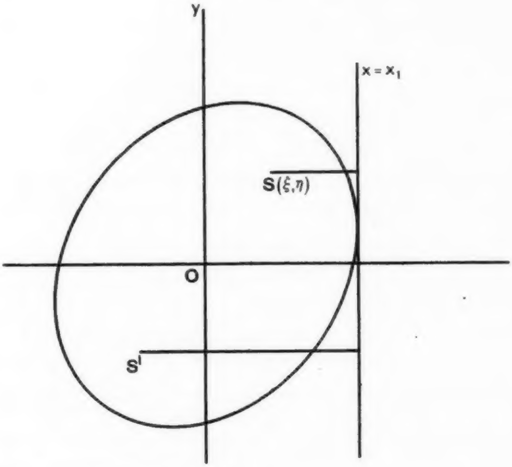
Thus $$x_1^2=b/(ab-h^2),$$

and similarly $y=y_1$ will touch (i) if

$$y_1^2=a/(ab-h^2).$$

Now if the focus S is (ξ, η), the focus S' is $(-\xi, -\eta)$.

Then $$(x_1-\xi)(x_1+\xi)=\text{square on a semi-axis},$$
and so $$x_1^2-\xi^2=y_1^2-\eta^2$$
$$=\text{square on a semi-axis}$$
$$=1/A \text{ or } 1/B,$$
where $$A+B=a+b, \quad AB=ab-h^2;$$
that is, A and B are the roots of
$$t^2-(a+b)t+(ab-h^2)=0.$$



The foci are therefore given by
$$\frac{b}{ab-h^2}-\xi^2=\frac{a}{ab-h^2}-\eta^2=\frac{1}{t}, \quad \text{..................(iii)}$$
where t is a root of the above quadratic.

It will be easily seen in a numerical case which value of t will make ξ and η real. Also in practice it is better to use the quadratic in $1/t$.

Lastly, the signs of ξ and η are determined by the sign of xy in equation (ii) above.

3. *Numerical example for the ellipse* $6x^2+4xy+8y^2=1$.

Here $$A+B=14, \quad AB=44.$$

The equation in t is
$$t^2-14t+44=0,$$
or $$44(1/t)^2-14(1/t)+1=0,$$
whence $$1/t=(7\pm\sqrt{5})/44.$$

The equations for the foci are therefore
$$\frac{8}{44}-\xi^2=\frac{6}{44}-\eta^2=\frac{7\pm\sqrt{5}}{44}.$$

Clearly the positive value of the radical will make ξ imaginary. Thus the real foci are given by

$$\frac{8}{44}-\xi^2=\frac{6}{44}-\eta^2=\frac{7-\sqrt{5}}{44}.$$

so that $\xi^2=(1+\sqrt{5})/44, \quad \eta^2=(\sqrt{5}-1)/44.$

Now from 2 (ii), $\tfrac{1}{2}\xi\eta=(\xi^2-\eta^2)/(6-8)$
$= -1/44,$

and hence ξ, η have opposite signs.

Thus the coordinates of the foci are

$$+\left(\frac{1+\sqrt{5}}{44}\right)^{\frac{1}{2}}, \quad -\left(\frac{\sqrt{5}-1}{44}\right)^{\frac{1}{2}};$$

and $$-\left(\frac{1+\sqrt{5}}{44}\right)^{\frac{1}{2}}, \quad +\left(\frac{\sqrt{5}-1}{44}\right)^{\frac{1}{2}}.$$

4. *Numerical example for the hyperbola* $3x^2+4xy-7y^2=1$.

Here $A+B=-4, \quad AB=-25.$

Thus $t^2+4t-25=0,$

and so $1/t=(2\pm\sqrt{29})/25.$

The equations giving the foci are

$$\frac{7}{25}-\xi^2=-\frac{3}{25}-\eta^2=\frac{2\pm\sqrt{29}}{25}.$$

Taking the negative sign to get positive values of ξ^2 and η^2, we get

$$-\frac{7}{25}+\xi^2=\frac{3}{25}+\eta^2=\frac{\sqrt{29}-2}{25}$$

and $\xi^2=(5+\sqrt{29})/25, \quad \eta^2=(-5+\sqrt{29})/25.$

Also $\tfrac{1}{2}\xi\eta=(\xi^2-\eta^2)/(3+7)=1/25,$

and so ξ and η have the same sign.

Thus the foci are

$$+\left(\frac{5+\sqrt{29}}{25}\right)^{\frac{1}{2}}, \quad +\left(\frac{-5+\sqrt{29}}{25}\right)^{\frac{1}{2}};$$

and $$-\left(\frac{5+\sqrt{29}}{25}\right)^{\frac{1}{2}}, \quad -\left(\frac{-5+\sqrt{29}}{25}\right)^{\frac{1}{2}}.$$

H. N. HASKELL.

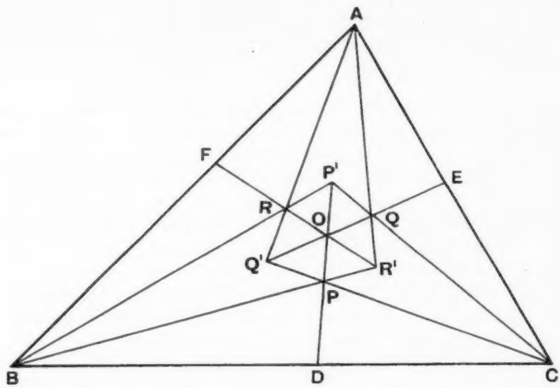
1476. *A theorem due to Professor F. Morley.*

Let P, Q, R be points interior to a triangle, such that

$$\angle CAQ=\angle QAR=\angle RAB=\tfrac{1}{3}A,$$
$$\angle ABR=\angle RBP=\angle PBC=\tfrac{1}{3}B,$$
$$\angle BCP=\angle PCQ=\angle QCA=\tfrac{1}{3}C.$$

To prove that PQR is an equilateral triangle.

Let CQ, BR meet in P', AR, CP meet in Q' and BP, AQ meet in R'.



1. Then, as the bisector of $\angle QAR$ is the bisector of $\angle BAC$, with similar statements at B, C, it is a theorem of elementary conics that QR', $Q'R$, RP', $R'P$, PQ', $P'Q$ are tangents of a conic.

Thus PP', QQ', RR' meet in a point; say, O. Let $P'OP$, $Q'OQ$, $R'OR$ meet BC, CA, AB respectively in D, E, F.

2. As BP, CP bisect $\angle P'BC$, $\angle P'CB$, therefore $P'P$ bisects $\angle BP'C$. So $Q'Q$, $R'R$ bisect $\angle CQ'A$ and $\angle AR'B$.

Whence, drawing perpendiculars to BP through B and to CP through C, which meet on PD, we see that $\angle BPD$, $\angle CPD$ are the respective complements of $\angle PCD$, $\angle PBD$, and the angle

$$\angle PDC = \tfrac{1}{2}\pi + \tfrac{1}{3}(B - C).$$

Similarly for CQA and ARB.

3. Hence $\angle P'QO$ and $\angle P'RO$, *i.e.*, respectively, $\angle CQE$, $\angle BRF$, being respectively the complements of the equal angles $\angle EAQ$, $\angle FAR$, are equal to one another. Thence, as $P'O$ bisects $\angle QP'R$, the triangles $P'QO$, $P'RO$ are images of one another in the line $P'O$. Hence QR is bisected at right angles by OP'. Similarly for RP and OQ', and for PQ and OR'.

4. By what precedes, $\angle AEO$ is $\tfrac{1}{2}\pi + \tfrac{1}{3}(C - A)$, and $\angle AFO$ is $\tfrac{1}{2}\pi + \tfrac{1}{3}(B - A)$. Whence, by the quadrangle $AEOF$, we have

$$\angle QOR = \pi - \tfrac{1}{3}(A + B + C) = \tfrac{2}{3}\pi.$$

Likewise $\angle ROP$ and $\angle POQ$. Hence PQR is an equilateral triangle.

This compares with Mr. J. M. Child's proof, *Math. Gazette*, October, 1922, and the proof by Richmond and Lob, *Proc. London Math. Soc.*,

ser. 2, vol. XXXI, referred to by Morley, in his volume, *Inversive Geometry*, 1933, p. 244. I am told that Mr. Lob has given a proof by projection from a cubic curve in three dimensions, which should be very interesting, and be the definitive proof: I have not seen this. The above, however, is an elementary metrical proof for the student of plane geometry.

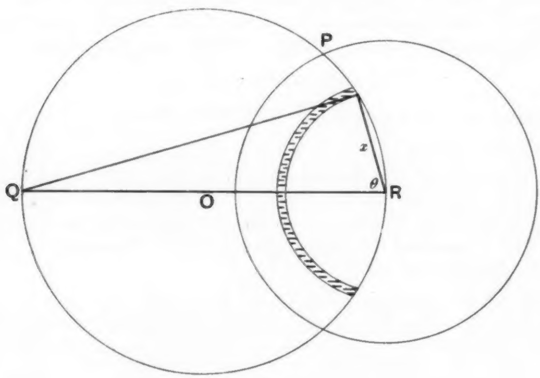
Postscript. It may be of interest to add that Steiner's proof of his theorem for the envelope of the pedal line of a triangle, contains the following result: "Let O, H, be the circumcentre and orthocentre of a triangle ABC. From A draw a line, on the side of AO on which AH lies, whose angular distance from AO is two-thirds of the angular distance of AH from AO. Similarly draw lines from B and C. These three lines form an equilateral triangle". (Steiner, *Werke*, ii, p. 641 (1856).)

Morley's theorem is derived from the common tangents of *cardioids*, and he distinguishes between the cardioid and (Steiner's) 3-cusped hypocycloid. But both these curves are quartic curves with 3 cusps, and one double tangent, easily transformable into one another.

H. F. Baker.

1477. *An old problem.*

The problem of the donkey tethered to the edge of a circular field appears to be of great interest to the student doing Newton's method for solving equations. The following method for obtaining the area involved may be of interest as being more in accordance with the work on calculus that usually precedes this stage of teaching.



The circular field of radius r has centre O. The donkey is tethered at R by a rope of length PR. Then in the figure $2r \cos \theta = x$, and if we denote the angle QRP by α, the area common to the two circles is

$$\int 2x\theta\, dx = -4r^2 \int_{\pi/2}^{\alpha} \theta \,.\, \sin 2\theta \,.\, d\theta$$
$$= 2r^2\{\alpha \cos 2\alpha - \tfrac{1}{2} \sin 2\alpha + \tfrac{1}{2}\pi\}.$$

If this area is to be the fraction k of the whole area πr^2 of the field, we obtain the usual equation

$$2\alpha \cos 2\alpha - \sin 2\alpha + \pi(1-k) = 0.$$

J. PEDOE.

1478. *The multiplication and division of decimals and contracted methods.*

I have read with interest Mr. Webb's article on the multiplication of decimals (*Gazette*, July, 1940) and I must admit that I was rather surprised to find that this particular item of the arithmetic syllabus should still cause so much controversy. I agree with Mr. Webb that in teaching arithmetic it should be our aim to select those methods which become automatic, which at the same time are accurate and which make as little demand upon the reasoning power as possible. The " Fractional method " which Mr. Webb advocates satisfies all these conditions and I am convinced that it is far superior to any other as regards accuracy. But I am more interested in the statement that the fractional method does not lend itself to contracted work.

As far back as 1927, when I had abandoned " Standard form " and other devices, I worked out a series of examples of contracted methods based on the fractional method. A few examples will make the method clear.

(i) Multiply 27·647289 by 0·006423 correct to three significant figures.

$$\text{The approximate answer} = 30 \times 0{\cdot}006$$
$$= 0{\cdot}180.$$

(I require 5 significant figures in my first partial product, therefore I write :)

$$\begin{array}{r|l}
2\cancel{7}\cancel{6}\cancel{4} & 7289 \\
6 & 423 \\
\hline
16588 & \\
1106 & \\
55 & \\
8 & \\
\hline
17757 & \\
\hline
\end{array}$$

(The number of decimal places to the left of the vertical line is 2 + 3, that is, 5.)

Therefore the answer is 0·178 correct to three significant figures.

(ii) Multiply 8·719426 by 0·47923 correct to two decimal places.

$$\text{The approximate answer} = 9 \times 0{\cdot}5$$
$$= 4{\cdot}5.$$

(Therefore we require the answer correct to three significant figures.)

$$\begin{array}{r|l}
\cancel{8}7\cancel{1}\cancel{9} & 426 \\
4 & 7923 \\
\hline
34878 & \\
6103 & \\
784 & \\
17 & \\
2 & \\
\hline
41784 & \\
\hline
\end{array}$$

(The number of decimal places to the left of the vertical line is $3+1$, that is, 4.)

The answer is 4·18 correct to two decimal places.

Experience has taught me to use the method known as " Equation of places " for division of decimals. The method is used for contracted work also. The main point to be observed is that the decimal point is ignored until the end.

Example. Divide 907·6569 by 21·3.

The approximate answer $=900 \div 20$
$=45.$

$$\begin{array}{rl}
 & 42613 \\
21\underline{3}\,) & 907\underline{6}569 \\
 & \underline{852} \\
 & 556 \\
 & \underline{426} \\
 & 1305 \\
 & \underline{1278} \\
 & 276 \\
 & \underline{213} \\
 & 639 \\
 & \underline{639} \\
 & \ldots
\end{array}$$

To find the position of the decimal point.

The last figure in the divisor is 3, one place to the right of the decimal point. The corresponding figure in the dividend is the first 6, one place to the right of the decimal point in the dividend. The decimal point in the quotient then comes after the figure 2; therefore the answer is 42·613.

Points in favour of this method are that it is accurate, automatic and easy to learn.

A few examples on division of decimals using contracted methods are appended.

(i) Divide 4·87643 by 32·41986 correct to three significant figures.

The approximate answer is $5/30 = 0{\cdot}17$.

(As we require the answer to three significant figures we correct the divisor to five significant figures, namely, 32420, ignoring the decimal point. We keep as many digits in the dividend as will allow of the first division, in this case five.)

$$
\begin{array}{r r|l}
 & 1 & 5041 \\
3\cancel{2}\cancel{4}\cancel{2}\underline{\cancel{0}}\ ) & 487\underline{6}4 & \\
 & 324\bar{2}0 & \\
\cline{2-2}
 & 16344 & \\
 & 16210 & \\
\cline{2-2}
 & 134 & \\
 & 130 & \\
\cline{2-2}
 & 4 & \\
 & 3 & \\
\cline{2-2}
 & \underline{1} & \\
\end{array}
$$

Underlining the last figure in the divisor and the corresponding figure in the dividend, the answer is found as 0·150.

(ii) Divide 0·843 by 112·642 correct to four decimal places.

The approximate answer is 0·008.

(We require the answer correct to two significant figures. Correct the divisor, if necessary, to four significant figures, that is, 1126. Add on a zero to the dividend to make the first division possible.)

$$
\begin{array}{r r|l}
 & 7 & 487 \\
1\cancel{1}\cancel{2}\underline{\cancel{6}}\ ) & \underline{8}430 & \\
 & \bar{7}882 & \\
\cline{2-2}
 & 548 & \\
 & 450 & \\
\cline{2-2}
 & 98 & \\
 & 90 & \\
\cline{2-2}
 & 8 & \\
 & 8 & \\
\cline{2-2}
 & \underline{\cdot} & \\
\end{array}
$$

The answer is 0·0075 correct to four decimal places.

The few examples I have chosen at random will, I think, make it clear that these methods which are based on straightforward multiplication and division of integers can be used for contracted work.

T. C. Batten.

1479. *A geometrical note.*

The article "Theorems on Perspectivity" * provides a striking example of the power and elegance of the methods advocated by Professor Baker † for dealing with problems of incidence geometry.

The first main theorem in the article (I, § 7) is well known: the

* *Mathematical Gazette*, XXIV (1940), 9-14.

† *Principles of Geometry*; in particular Vol. I (Cambridge 1922), pp. 70 *et seq.*

configuration is that of desmic tetrahedra *, which is, in fact, none other than the projective configuration defined by a cube.

The second main theorem can be disposed of quite simply by constructing the following syzygies. Using for the points the symbols employed in the article quoted (but replacing O by Φ to avoid confusion with zero), write the syzygies that express that $ABCD$, $A_1B_1C_1D_1$ are in perspective from Φ in the form :

$$A_1 = A + \Phi, \quad B_1 = B + \Phi, \quad C_1 = C + \Phi, \quad D_1 = D + \Phi,$$

where there is a single syzygetic relation, which need not be specified, † connecting the points Φ, A, B, C, D. Then the symbols for the other points concerned are :

$$\begin{array}{lll}
 & BC = B - C = B_1 - C_1, \text{ etc.} & U = AB - CD = AC - BD, \text{ etc.} \\
i.e., & U = A - B - C + D & \alpha = -A + B + C + D \\
 & V = -A + B - C + D & \beta = A - B + C + D \\
 & W = -A - B + C + D & \gamma = A + B - C + D \\
 & \Omega = A + B + C + D & \delta = A + B + C - D. \\
 & \alpha_1 = -A_1 + B_1 + C_1 + D_1 = \alpha + 2\Phi, \text{ etc.} & \\
 & \Omega_1 = A_1 + B_1 + C_1 + D_1 = \Omega + 4\Phi. &
\end{array}$$

Thus Ω, Ω_1, and Φ are collinear, and each of the four pairs, α, α_1 are collinear with Φ.

T. G. Room.

1480. *On Note* 1429 (*Gazette*, XXIII, p. 471).

The problem of finding a real fraction

$$y = (ax^2 + bx + c)/(a'x^2 + b'x + c'), \quad \text{..................(i)}$$

with given bounds, may be solved in another way. Suppose that y is not to lie outside specified bounds α, β for real x, and is to take these values at $x = x_1$, $x = x_2$ respectively. Then we may take

$$\frac{y - \alpha}{y - \beta} = \lambda \left(\frac{x - x_1}{x - x_2} \right)^2,$$

where $\lambda < 0$; for then it is obvious that $y - \alpha$, $y - \beta$ must have opposite signs so that y is in general between α and β, and $y = \alpha$, $y = \beta$ at $x = x_1$, $x = x_2$ respectively.

If we want a fraction whose value cannot lie between α and β, we have only to take $\lambda > 0$.

One of the examples in Note 1429 requires $2 \leqslant y \leqslant 5$. Suppose y assumes the values 2, 5 at $x = -1$, 2 respectively. Then the fraction is found from

$$\frac{y - 2}{y - 5} = \lambda \left(\frac{x + 1}{x - 2} \right)^2,$$

where we take $\lambda = -\frac{1}{2}$.

* See, *e.g.*, *Principles of Geometry*, Vol. II, p. 213, and Hudson, *Kummer's Quartic Surface* (Cambridge, 1905), pp. 1-3.

† That this syzygy is not used implies that the configuration is properly one belonging to space of four dimensions.

The other example, in which $\alpha=2$, $\beta=5$, $x_1=0$, $x_2=-3$, is given by

$$\frac{y-2}{y-5}=\lambda\left(\frac{x}{x+3}\right)^2,$$

where λ is taken to be 4.

This method covers all cases except that in which y can take any value whatever.

There is a slight mis-statement in Note 1429. If the equation (i) is solved for x, we get real values of x if

$$(b'^2-4a'c')y^2+2y(2a'c+2ac'-bb')+(b^2-4ac)\geqslant 0,$$

or

$$Ay^2+2By+C\geqslant 0.$$

If $B^2-AC<0$ and $A>0$ there is no restriction on y.

If $B^2-AC<0$, A cannot be negative, as is proved in Chrystal's *Algebra*.

G. W. BREWSTER.

1481. *Some algebraic computation.*

There is a race of arithmeticians set apart from the rest of mankind who delight in computation for its own sake, spending a vast number of peaceful hours, for example, in evaluating various functions to many hundreds of decimal places, in evaluating

$$2^{1,111111,111111,111110} \pmod{1, 111111, 111111, 111111}$$

to establish the primality of the modulus, or in discovering and testing that

6049382716049382717 6049382716049382717
3950617283950617284 3950617283950617284

of the digital form $AABB$, is a square number, being, in fact, the square of

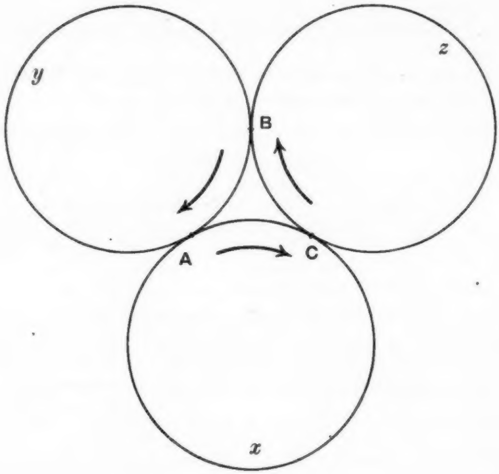
777777, 777777, 777777, 8, 777777, 777777, 777777, 8.

(This last may be a new result.)

The parallel pursuit of computing at leisurely and prodigious length with letters may be equally well known, but I confess that I have not seen any specific name given to the business nor have I encountered many published results of such activities. This may be due solely to my ignorance, and I apologise in advance if that be the case. The purpose of this note is to report a few results which seem to be of considerable elegance in themselves, and which may have the further attraction of being unfinished in the most difficult case, so that they offer themselves as a challenge to those whose tastes incline to such work.

A fruitful source of algebraic computation is the calculation of the various quantities involved in the periodicity of some system of moving points. My present investigation began in a classic problem of this kind, introduced to me many years ago as the " Seventh

Devil ", from an examination paper of St. John's College, June 1851, concerned with the motions of three men in concentric circles. A few slight alterations in that question produced quite a different one as follows :



Three motor cars (in view of the courses to be run it would be inhuman to have them men) run uniformly and clockwise in equal circular tracks, which are tangent to one another as in the figure, starting at the points of contact, A, B, C, and completing one circuit of their track in x, y, and z seconds respectively. Whenever two cars pass exactly at A, B, or C, they interchange speeds. Required to evaluate such features of the period of motion, until the initial position recurs, as the cycle of interchanges of speed, the total time, and the number of circuits of each car.

To eliminate endless fractions in the arithmetic, I examined only a few particular sets of values of x, y, and z, and was surprised to get widely different results from closely similar values.

I. $x=6 \,.\, 6k$; $y=6(6k+1)$; $z=6(6k+2)$.

To obtain actual interchanges of speed in this case, it is *necessary* to take $k=2$, reducing the analysis to a single problem, mainly arithmetical in character.

For 72, 78, 84 seconds, the period has 40 interchanges : CA, $BC\ BC$, $CA\ CA$, $BC\ BC$, $(CA\ CA)_7$, $BC\ BC$, $CA\ CA$, $BC\ BC$, $CA\ CA$, $BC\ BC$, $CA\ CA$, $BC\ BC$, $CA\ CA$, $BC\ BC$, CA, where the suffix indicates repetitions of the pair of changes within the brackets, a notation used throughout.

Time $=19656$ secs., and the numbers of circuits are, A : 247 ; B: 259 ; and C: 251.

This case revealed little of what was to come.

II. $x=6(6k-2)$; $y=6(6k-1)$; $z=6 \, . \, 6k$.

The general analysis was particularly confusing until it was discovered that there are two distinct subdivisions.

(i) k *odd* : The cycle of interchanges in the period is

$$(BC,\ AB,\ AB,\ BC)_{(3k-1)/2}.$$

$$\begin{aligned} &\text{Time} && =36k(3k-1)(6k-1) \text{ secs.} \\ &\text{Circuits,} && A=\tfrac{1}{2}(36k^2-9k+1), \\ & && B=6k(3k-1), \\ & && C=\tfrac{1}{2}(3k-1)(12k-1). \end{aligned}$$

(ii) k *even* : The cycle of interchanges is

$$(BC,\ AB,\ AB,\ BC)_{(3k-2)/2},\ BC,\ (AB,\ AB)_{3k+1},$$
$$BC,\ (BC,\ AB,\ AB,\ BC)_{(3k-2)/2}.$$

$$\begin{aligned} &\text{Time} && =108k(6k-1)(3k-1) \text{ secs.} \\ &\text{Circuits,} && A=(6k-1)(9k-1), \\ & && B=54k^2-18k+1, \\ & && C=(3k-1)(18k-1). \end{aligned}$$

The analysis involves the solution of numerous indeterminate equations, the examination of a delicate sequence of inequalities, and of course various summations of simple series.

III. $x=6(6k-1)$; $y=6 \, . \, 6k$; $z=6(6k+1)$.

In case II, although the period lengthens as k increases, the alterations are of a systematic character that generalises readily. But in the present case, there is a species of infinite growth that seems quite unmanageable. The form of the early part of the cycle is as follows :

$k=1$: AB, BC BC, **AB** ...

2 : AB, BC BC, CA CA, **BC** ...

3 : AB, BC BC, CA CA, AB AB, **CA** ...

4 : AB, BC BC, CA CA, AB AB, BC BC, **AB** ...

5 : AB, BC BC, CA CA, AB AB, BC BC, CA CA, **BC** ...

and so on.

Thus, while the 4th interchange is AB for $k=1$, it remains CA ever afterwards for $k>1$. A similar phenomenon occurs at the 6th, 8th, 10th, ... interchange, the lowest value of k for each "irregularity" giving a different result from all subsequent values.

The first two complete cycles are :

$k=1$: AB, $(BC, BC, AB, AB)_2$, $CA\ CA$, $BC\ BC$, $AB\ AB$, $BC\ BC$, AB.

$k=2$: AB, $(BC, BC, CA, CA, BC, BC, AB, AB)_2$, $(CA, CA, AB, AB, BC, BC, AB, AB)_2$, $CA\ CA$, $AB\ AB$, $BC\ BC$, $AB\ AB$, $CA\ CA$, $BC\ BC$, $CA\ CA$, $AB\ AB$, $CA\ CA$, $BC\ BC$, AB.

These results throw little light on the general form of the cycle, and the rapid growth in length induced a proportionate feeling that it was time to hand on the question to some other lover of algebraic computation.

As they may be of some small service in checking a general formula, I add the other values for :

		$k=1$	$k=2$
Time		$=2520$	30888
Circuits,	$A=$	73	432
	$B=$	72	440
	$C=$	69	421

T. R. Dawson.

1482. *Descendants of Sir William Rowan Hamilton.*

In the July *Gazette* (XXIV, No. 260, p. 158) Professor E. T. Whittaker remarks that " it would be of interest to ascertain whether Hamilton has any living descendants ", and mentions his one grandchild, John Rowan Hamilton O'Regan. Mr. A. Robson and Mr. F. Puryer White have supplied the following information.

John Rowan Hamilton O'Regan, son of the Ven. John O'Regan, Archdeacon of Kildare, went up as a scholar to Balliol College, Oxford, from Clifton in 1889 ; he was a hockey blue. For about twenty-seven years from 1894 he was a master at Marlborough College, and died at Marlborough, 1st July, 1922.

He married, in 1912, Phoebe Alice, daughter of the Rev. J. T. H. Abbott, of Creenahoe, Co. Fermanagh, and had three sons : his eldest son, John William Hamilton O'Regan, entered the Ceylon Civil Service in 1935 and is private secretary to the Governor of Ceylon ; his second son, Michael Lionel Victor Rowan Hamilton O'Regan, is a Second Lieutenant of the Royal Ulster Rifles ; his third son, Patrick Valentine William Rowan Hamilton O'Regan, is a Postmaster of Merton College, Oxford, and is at present waiting to join the forces.

Mrs. O'Regan still lives in the house (Killycoonagh, Marlborough) which John Rowan Hamilton O'Regan had built on the model of a Roman villa.

1329. " You see those bricks? Well, the whole story's written there. Each cube has four sides. I've filled three. If I turn them this way—they fit, and Mr. Furness is pretty nearly caught."—Harriette R. Campbell, *The String Glove Mystery.* [Per Mr. C. A. B. Smith.]

REVIEWS.

ELEMENTARY NUMBER THEORY.

1. **Modern elementary theory of numbers.** By L. E. Dickson. Pp. vii, 309. 18s. 1939. (University of Chicago Press; Cambridge University Press)

2. **Elementary number theory.** By J. V. Uspensky and M. A. Heaslet. Pp. x, 484. 26s. 1939. (McGraw-Hill)

3. **First course in theory of numbers.** By H. N. Wright. Pp. vii, 108. 12s. 1939. (John Wiley, New York; Chapman and Hall)

It is not so many years ago that devotees of number theory considered their subject to be the neglected Cinderella of the mathematical world. True though this may have been in the past, it certainly no longer holds, judged either by the importance and quantity of research, or by its position as a subject of study in a large number of Universities, or by the number of books appearing upon every aspect of the subject, and in particular upon elementary number theory.

The conception of what is elementary differs with each writer. Nearly all would exclude applications of complex variable theory. Most would be inclined to omit lengthy and detailed investigations more fittingly left to research journals or junior doctorate theses. The geometry of numbers, or Diophantine approximation, apart from classic results mostly associated with continued fractions, is still an uncommon feature of elementary books despite the simplicity, the generality and the richness of application of some of the results. Some writers would exclude, for one reason or another, even a short account of the classic theory of binary quadratic forms, though they may often include results which could be proved more easily and with greater generality by this theory. Practically all writers give an account of Diophantine equations depending of course upon the previous preparation given in the particular book.

Some desiderata for the style of elementary books naturally suggest themselves. Definitions and notation should be given prominence and any departure from usual conventions should be emphasized. Theorems should be carefully enunciated, and given as self-contained statements when possible, though occasionally not when this would require great length. Proofs should leave little to the imagination of the reader, since what is obvious with further experience may be obscure at first. Emphasis, whenever possible, should be laid upon the essential point of a proof. Then spacing, the setting out of formulae, and printing, should be as attractive as possible, especially when long and complicated proofs are involved. Economy in space may cost the reader a great deal of mental discomfort; he ought always to feel that he is being conducted gently when on rough and unfamiliar ground.

Professor Dickson may be confidently expected to strike out a line of his own in any book that he writes. Thus the present one consists of quite a number of distinct sections which no one else would think of bringing together. There is first a fairly complete account of really elementary number theory, including binary quadratic forms, and also ternary and other quadratic forms, with perhaps too much emphasis upon representation of numbers by special forms. There is an account of various indeterminate equations including the representation of zero by ternary and quaternary quadratic forms and others. Results are given related to Waring's problem, namely on the representation of numbers by particular polynomial summands, many of the results being due

to Dickson and his school. Finally there is an appendix proving in full detail Dirichlet's theorem on the infinity of primes in an arithmetical progression.

This brief sketch of the contents shows that there is considerable variation in the knowledge and aptitude expected from the reader in different sections. In a number of places, numerical illustrations are given which assume little intelligence or great naïveté on the reader's part; for example, the division process exemplified by $k=mq+r$, $0 \leqslant r < m$, is prefaced by $64=10 \times 6+4$. The statement, however, beginning Chapter IV, that "the problem to find all integral solutions of $x^2+y^2=41$ is equivalent to that for $X^2+4XY+5Y^2=41$ which is derived from the first equation by the transformation $x=X+2Y$, $y=Y$" is on a rather different footing, requiring just the slightest proof, namely, of the 1-1 correspondence between integer sets of x, y and X, Y. In many chapters, however, considerable competence, maturity and concentration are required from the reader.

Dickson's style is rather terse but he can be expected to take notice of pitfalls and snares awaiting writers; for example, he defines properly the multiple roots of a congruence. It would, however, help the reader if now and then he amplified his statements. Thus in stating that a number m is represented by a quadratic form $f(x, y)$ if integers x, y exist such that $f(x, y)=m$, it would be as well to say that a representation of 0 by (0, 0) is a trivial one since this plays no part in the theory and saves a moment's thought in reading the definition that a universal indefinite ternary quadratic form is one which represents all integers (the trivial representation of 0 being allowed). Again, theorem 80 states that "any ternary quadratic form $F(x, y, z)$ with rational coefficients can be expressed as $g_1X^2+g_2Y^2+g_3Z^2$ where each g_i is a rational number, while X, Y, Z are independent linear homogeneous functions of x, y, z with rational coefficients whose determinant is not zero". This statement is quite true, but it would be as well to distinguish the cases when the determinant of F is zero as then some of the g's may be zero and the corresponding X, etc., are really arbitrary. He states without comment that an integral ternary form has a minimum when he might well give even more than a reference to the binary case which he proves in an earlier chapter.

Most of the book is strongly coloured by Dickson's original work. Many readers would probably feel happier if some of the results were given with less specialisation, as even one particular result may be sufficient to give some idea of the nature and scope of a method. But one must allow for the enthusiasm of an investigator, even if one cannot accept the statement that "The following result is used more often than any other in researches in the theory of numbers. Theorem 84. The positive integers which are not sums of three squares are exclusively those of the form $4^k(8n+7)$, where $k \geqslant 0$".

But whatever be the reader's degree of knowledge he is sure to find results, examples and tables of interest, importance and suggestiveness.

The book by Uspensky and Heaslet is really an elementary book comprising the usual results up to and including the theory of quadratic residues. Some developments of these not usually found in elementary books are given, for example, Lamé's theorem that the number of divisions required to find the greatest common divisor of two numbers is never greater than five times the number of digits in the smaller number; or again, results on the least remainder algorithm, *i.e.* when negative remainders are admitted in division; Bonse's proof that if p_n is the nth prime, $p^2_{n+1} < p_1 p_2 \cdots p_n$; and Meissel's practical formula for calculating the number of primes less than a given number. A few of the chapters have practical applications, sometimes as appendices, for example, to Nim, Magic Squares, Calendar Problems and Card Shuffling.

Then there are chapters on Bernouillian numbers and the methods of Liouville. The authors, in order to limit the size of the book, which is fairly large, containing nearly 500 pages, have deliberately omitted a number of topics, for example, continued fractions, geometric and analytic methods.

The usual theory of quadratic forms is not given but there is an interesting chapter (XI) on some problems connected with quadratic forms. The following important result is given. Let a be a non-square integer and N, m integers such that

$$N^2 \equiv a \pmod{m}.$$

Then for some integer λ with $\lambda^2 < 4a/3$, integers x, y exist satisfying

$$\lambda m = x^2 - ay^2, \quad x \equiv Ny \pmod{m}, \quad (x, y) | (\lambda, m),$$

where (x, y) denotes the highest common factor of x, y. Applications are given to the Diophantine equation

$$x^2 - ay^2 = m, \quad \text{for} \quad a = -1, \ -2, \ -3, \ -5, \ -7\ ;$$

and also for some positive values of a after an account of the equation $t^2 - au^2 = 1$. This equation is also used to give Kummer's proof of the law of quadratic reciprocity. The chapter closes with a proof that every positive integer is a sum of four squares.

The chapter on Diophantine equations contains much novel and interesting material, especially on equations of the form $x^2 + ay^2 = z^n$ for particular values of a for which the congruence result above is very useful. Then the equation $y^3 = x^2 + c$ is considered for a few values of c. It is proved that when $c = 4$, 2 the only integer values of y are 2 and 3 respectively. There is also given A. Brauer's ingenious proof that when $c = -2$ the only integer solutions are given by $x = \pm 1$, $y = -1$. I find it surprising that the authors do not deal with cases such as $c = -7$, where it is easily shown that there are no integral solutions. Thus y cannot be even or of the form $4n - 1$, since no squares are of the forms $4n + 2$, $4n + 3$. Also, from $x^2 + 1 = (y + 2)(y^2 - 2y + 4)$, y cannot be of the form $4n + 1$ since then $y + 2$ is positive and of the form $4n + 3$ and so cannot divide $x^2 + 1$. This proves the result.

It is stated that Delaunay was able to show that the only solutions of the equation $x^2 - 17 = y^3$ with $x > 0$ are

$$\begin{array}{rrrrrrrrr} x = & 3, & 4, & 5, & 9, & 23, & 282, & 375, & 378661, \\ y = & -2, & -1, & 2, & 4, & 8, & 43, & 52, & 5234. \end{array}$$

I think that these solutions were first given by me in *Science Progress*, vol. 18, 1923, and that Nagell first proved in 1930, in the *Proceedings of the Norwegian Academy* at Oslo, that these solutions were the only ones. The chapter also contains illustrations of the method of infinite descent applied to particular problems, for example, $x^4 + y^4 = z^2$, $x^3 + y^3 = z^3$ and to other numerical problems. We find, for example, the first three solutions in integers of the old equation of the Arabs,

$$x^2 + 5y^2 = z^2, \quad x^2 - 5y^2 = t^2,$$

the third set being numbers of fifteen digits.

The chapter on Liouville's methods is the last and contains a long but essentially elementary proof that every positive integer except those of the form $4^k(8N + 7)$ is the sum of three squares.

The authors have unquestionably produced an interesting, readable book eminently suitable for beginners. Readers will find it a pleasant introduction to number theory, for the proofs are well amplified, though occasionally long drawn out, and the student is not led away too hastily. He will be stimulated

by the historical notes and details, often given at great length. He will be teased by a number of the problems designed to test his grasp of the theory, for nowhere else as in number theory are problems so easy when the solution is known and so difficult when not.

Wright's book is, as its title indicates, a small primer on number theory. Its five chapters are on divisibility, simple continued fractions, congruences, quadratic residues and Diophantine equations, in this order. The lengthy chapter on continued fractions, which also contains an account of periodic continued fractions and the Diophantine equation $x^2-Dy^2=N$, might well have been placed at the end of the book.

As the book is expensive for its size, one can reasonably expect the author to make out a strong case for its appearance. He may do this either by originality in presentation or in choice of material, both obviously very difficult for the bare elements of a classic subject—in fact, the only authors referred to in the index are Dickson, Diophantus, Euler, Fermat, Gauss, Jacobi, Kummer, Lehmer D. N., Pythagoras and Wilson—or perhaps by clarity of treatment. I do not think that he has been altogether successful. There is occasionally lacking a few words in the proof or something in the method of presentation so that emphasis is not given to the essential point of a proof—for example, in his proof of the form of even perfect numbers as contrasted with those of the other two books.

Some of the results in the chapter on congruence are not stated as precisely as they should be; for example, Theorem 18 is " A congruence of degree n, in one unknown with a prime modulus, whose coefficients are not all zero, has at most n distinct roots ". The author has not defined what he means by a congruence of degree n but his proof shows that he takes the coefficient of the highest power of x, namely x^n, to be not divisible by p, the prime modulus. Further, the enunciation should read that not all the coefficients are divisible by the prime modulus. Then a little later on he refers to equal roots of a congruence though the term has not been defined.

The book is well printed and pleasing to the eye, and the theorems are well set out, making it an easy book to read. Some misprints occur in places where they might obviously be expected, for example, in the writing of a continued fraction on p. 28, and in the complicated indices of some powers on p. 68, lines 6 and 8. L. J. MORDELL.

Development of the Minkowski geometry of numbers. By H. HANCOCK. Pp. xxiv, 839. 60s. 1939. (Macmillan)

The author states that the purpose of this book is to give a complete exposition of all that Minkowski did under the subject entitled Geometry of Numbers; and that he proposes to develop the treatment in such a way that the book is not a mere compendium of isolated results in number theory. It will prove very useful to many readers to have all these results gathered into one volume, and more especially to those who have difficulties with the German language. These difficulties are aggravated by Minkowski's style, which is not an easy one for the reader. The author, with the help of his students at the University of Cincinnati, has revised and sometimes amplified Minkowski's proofs, corrected some errors, and has had new figures drawn. All of this should prove of material assistance to his readers.

It is to be regretted that considerations of space have led Professor Hancock to confine his account almost entirely to the work done by Minkowski. An additional chapter giving only the briefest account of the recent new and important work on the geometry of numbers would have added to the value of the work. L. J. MORDELL.

The Theory of Group Characters and Matrix Representations of Groups. By DUDLEY E. LITTLEWOOD. Pp. viii, 292. 20s. 1940. (Clarendon Press, Oxford)

One of the difficulties which a student of the theory of group characters encounters when reading the original memoirs is due to the fact that the subject is introduced by various authors from apparently quite different angles. The three most important lines of approach are as follows: (*a*) the point of view of pure group theory, in which the group elements are specified solely by their multiplication table; this was the way in which Frobenius first defined the group characters; (*b*) the theory of hypercomplex numbers (algebras) and more particularly the theory of substitutional analysis (A. Young); (*c*) the theory of matrix representations, in which the characters are defined as the spurs of the matrices which represent the group elements.

This last-mentioned method has proved most successful for developing the theory, and since I. Schur's classical paper of 1905 has been employed by most writers, including the author, to define the group characters.

It is one of the merits of this book that the reader is made familiar with the foundations of all three methods, the link between (*a*) and (*b*) being established by a study of the Frobenius Algebra (Chapter IV), whilst the connection between (*b*) and (*c*) is demonstrated with the help of certain theorems on Algebras (Chapter II). The book also contains brief introductory chapters on Matrices and on Groups (Chapters I and III).

About one-third of the book (Chapters V-VIII) is concerned with the general symmetric group of order $n!$. Frobenius's celebrated formula for the characters of the symmetric group is derived and a brief account is given of A. Young's theory of tableaux. The theory of *immanants* (Chapter VI), which was developed by the author in collaboration with A. R. Richardson, rovides a powerful tool for the investigation of the characters of the symmetric group. If $A=[a_{rs}]$ be an arbitrary matrix and $\chi^{(\lambda)}(S)$ a simple character of the symmetric group, the corresponding immanant is defined as

$$| a_{rs} |^{(\lambda)} = \sum_s \chi^{(\lambda)}(S) a_{1e_1} a_{2e_2} \dots a_{ne_n},$$

where the summation is extended over all permutations $S: i \to e_i$. The determinant and permanant are particular cases of immanants. The S-functions, named after I. Schur, are immanants belonging to a special matrix of fundamental importance. Various generating functions of S-functions are obtained and their values are computed for certain cases.

In Chapter IX the author returns to the general discussion of finite groups and illustrates how the table of characters can be used to investigate the structure of the group. This application of the theory is particularly interesting and suggests further research.

The last two chapters of the book deal with continuous groups of matrices. They contain an account of the *invariant matrices* associated with an arbitrary matrix A (Chapter X). The classical canonical forms of these matrices are established and an application to invariant theory is sketched. Finally the group of *unitary matrices* (Chapter XI) is studied in detail, and after the concept of integration over the group manifold has been introduced, the characters of the unitary and orthogonal groups are derived.

In the preface the author comments on the scarcity of textbooks on the subject and states that the purpose in writing his book was " to give a simple and self-contained exposition of the theory. . . ." It is also claimed that " no specialised knowledge is required of the reader beyond that obtained in an ordinary degree course in mathematics ". The reviewer cannot say that

this particular aim has been achieved. In common with many modern mathematical publications this book is written in a style which is generally too concise to make the subject intelligible to a beginner, and even an advanced student will occasionally find it difficult to follow arguments and proofs with which he is already acquainted. It is of course realised that so complex a subject will not be readily mastered by anybody and that it is impossible in a book of reasonable size to exhibit each step of an intricate algebraical manipulation. Our criticism is rather directed against the habit of leaving gaps in logical deductions without due warning, which confuses the reader and spoils his pleasure in reading the book. To mention only one instance: on p. 179 the author defines the pth induced matrix $A^{[p]}$ of a matrix A and continues: " Quite clearly, if $AB=C$, then $A^{[p]}B^{[p]}=C^{[p]}$." Now it seems to the reviewer that a person who is not already familiar with this theorem—and such a person is likely to be a beginner in the subject—will not find this step " quite clear " and may well be held up by trying without assistance to bridge the gap whose width he has been led to underestimate. There are unfortunately a great number of such logical discontinuities in the text, some of which, it is true, are compensated for by page references and an extensive bibliography.

The printing is very well done and the tables of group characters compiled in the Appendix should prove of value to research workers, to whom this book will primarily be of interest. W. L.

New Method Arithmetics. By J. Murray. Pupils' Bk. I. Pp. 95. 1s. 4d. Teachers' Bk. I. Pp. 64. 2s. 6d. Pupils' Bk. II. Pp. 95. 1s. 6d. Teachers' Bk. II. Pp. 62. 2s. 9d. 1940. (University of London Press)

The aim of the pupils' books is to provide a suitable course of work in number for children of seven and eight years of age respectively. Book I devotes itself to the addition and subtraction of number and money, while Book II deals with simple multiplication and division of number, simple fractions, telling time, the measurement of length together with the revision and extension of the work in Book I. In addition to supplying answers the teachers' books give further examples for extra practice and help and guidance in putting the scheme of work into operation.

The author has broken away from the traditional methods of number teaching and has developed his work along the most modern lines. The old-fashioned serial memorisation of the various Tables does not take into consideration the fact that different number combinations present varying degrees of difficulty. It is a pleasing feature of this work to observe that a proportionate distribution of practice is given in accordance with the ascertained difficulty of the various number facts. Until recent years little or even no attention was given to the zero facts. Miss Margaret Drummond in her book *The Psychology and Teaching of Number* (Harrap) makes no reference at all to these facts. My own experience, which seems to be borne out by Dr. F. J. Schonell's researches recorded in *Diagnosis of Individual Difficulties in Arithmetic* (Oliver & Boyd), indicates that where such work is not taken the progress of the pupil is likely to be less smooth and less certain. With young children the zero combinations are very important. The New Method Arithmetics give satisfactory attention to the zero facts.

The general planning of the books is good. The work is carefully graded and the pupil is led on through the less difficult number facts first, each difficulty being met with a correspondingly correct number of practices. The practices can be used repeatedly for revision or as diagnostic tests and remedial exercises. One of the features of the pupils' books is that a certain

amount of space is devoted to explanations. With immature children and an overcrowded syllabus many teachers would prefer to make their own explanations; but, if they serve no other, the explanations in the book serve an excellent purpose if they prevent the teacher from sidetracking and from omitting any of the small but essential points arising out of the work.

Problem work is not overstressed and one of its good points is the varying form of the cues which are used. The following are examples of cues used in problems on addition: What is the total number? How many altogether? How many in all? How many do the numbers make? How many were counted?

In dealing with column addition the method of adding downwards is recommended and in subtraction the method of equal additions is suggested. No mention of the expressions Short Division and Long Division is made in Book II and the method of Long Division, in which the quotient is written above the dividend, is used throughout.

The author has shown himself to be well acquainted with modern research work in arithmetic, and has made the best possible use of its findings.

H. WEBB.

Simple Algebra. By M. MILFORD and R. C. LYNESS. Pp. 199. 3s. 6d.; without answers, 3s. 1940. (Arnold)

Certain " educational psychologists " have held that in Mathematical work the learner should never be shown anything incorrect, presumably lest he remember the wrong thing in preference to the right. The authors of this excellent little book take the opposite view. Sets of exercises headed " the following are either true or false; which? " are deliberately included, " in the hope that practice in noticing errors may help the reader in finding mistakes in his own work ". On the assumption that these exercises are studied in class with suitable discussion, there appears much to commend them.

Whether the book is suitable for individual study with each to " go at his own pace and at the same time learn to teach himself to some extent " (as the preface suggests is possible) is more open to conjecture. The same doubts, of course, apply to any work designed for this standard, namely as an ordinary first course.

This particular book ends with work on Simultaneous (Linear) equations, quadratic equations being nowhere mentioned, although indices are clearly explained. The word " clear " is indeed one which comes to mind continually while studying the text. The exercises, also, are good, and numerous sets of test and revision papers are included. (Answers to these papers are on perforated leaves, while an edition is available without answers.)

It is pleasing to see examples involving other portions of the Mathematics course (*e.g.* equations based on the relations between angles of triangles, etc.), and others on everyday affairs such as toast-racks, lengths of cinema programmes, and the amount of satisfaction to be obtained by eating successive bars of chocolate.

Two other points are worthy of individual mention: the questions and diagrams in the chapter on " First Graphs ", and the use of the word " puzzles " as a heading to a set of problems. These are the very things to interest a beginner.

Portions of the book not already mentioned deal with Letters for Numbers, Formulae, Simple Equations, Manipulation, and Directed Numbers. Books dealing with this type of work are often of a high standard. Even so, the Senior Mathematics Masters of Repton and Bristol Grammar Schools have produced one better than most.

F. W. K.

Sur les ensembles de distances des ensembles de points d'un espace euclidien. By Sophie Piccard. Pp. 212. 7.50 Swiss francs. 1940. Mémoires de l'Université de Neuchâtel, tome XIII. (Gauthier-Villars)

Let E be a sub-set of a Euclidean n-space R^n. The set of all non-negative numbers each of which is the distance between at least one pair of points of E is called the set of distances of E and is denoted by $D(E)$. For instance, when E is the set of vertices of the regular unit simplex in R^n, $D(E)$ consists of the two numbers 0 and 1, while if E is the segment [01] then $D(E)=E$.

Properties of the sets $D(E)$ corresponding to different classes of sets E have been discussed in isolated papers, mainly by Polish mathematicians. Professor Piccard gives an account of these results as well as a detailed account of her own painstaking researches in this field. A notable feature of this book is the widely varying techniques required in the various chapters; nevertheless, a large part of it requires only the most elementary mathematics.

The book is divided into four chapters. In the first, general properties of $D(E)$ are discussed. It is shown, for instance, that if $E \subset R^1$ has positive measure then $D(E)$ contains a segment $[0\delta]$ and that, although $D(E)$ need not be measurable when E is, it is true that if E is analytic so is $D(E)$. The following result is established by elementary means. If A and B are two finite sub-sets of R^1 (each being such that no distinct pairs of its points have the same distance apart) then, in order that $D(A)=D(B)$, it is necessary and sufficient that A and B should be congruent.

The second chapter is concerned with the properties of $D(E)$ when E is congruent with its complement. Assuming the Continuum Hypothesis, the existence of certain "singular" sets of this type is established.

The third chapter deals with $D(E)$ when E is perfect. Apart from a few general results, *e.g.* that $D(E)$ is perfect when E is, this chapter contains an account of numerical properties of the set of distances of the Cantor Middle Third set, π, and its generalisations (when we use a scale of notation different from 3). It is shown, for instance, that $D(\pi)=[01]$ although π has measure zero.

In the last chapter the problem of characterising the sets $D(E)$ is taken up. It is shown that any finite set (but not every countable set) of non-negative numbers which includes 0 is the set of distances of *some* set E in R^n for *some* n. Sets of $2n$ positive numbers are given which, together with 0, cannot be the set of distances of any sub-set of R^n. The following example illustrates some of these results. There is no set $E \subset R^2$ such that $D(E)=\{0, 1, 2, 4\}$, but the vertices of a tetrahedron whose base is a triangle with sides 1, 2, 2 and whose other edges are all 4 form a set E with $D(E)=\{0, 1, 2, 4\}$.

In certain parts of this book there are obvious points of contact with the work of Menger and his pupils on Metric Geometry, which has been so admirably summarised in the report of L. M. Blumenthal (reviewed in this *Gazette*, XXII (1938), 516) and it is surprising to find no mention of their work.

John Todd.

The Masses of the Stars. H. N. Russell and C. E. Moore. Pp. 236, viii, 53 tables. Paper, 21s. 1940. (University of Chicago Press; Cambridge University Press)

Current theories of the evolution of the universe are based largely on our knowledge of stellar masses. These range from about a tenth to hundreds of times the mass of the sun, and the relations of a star's mass to its luminosity, to its rate of energy production, and to its evolutionary stage are amongst the most fascinating problems of modern astrophysics. The work under review presents substantially all the direct evidence regarding the masses of the stars which is at present available.

Gravitational motions in binary systems can be observed visually and spectroscopically, as well as inferred from the light variations of eclipsing pairs. From such observations springs all our knowledge of stellar masses. The principle of the methods used is so simple that the mathematician might be excused if he dismisses it as Keplerian with no more than a cursory glance. Not the least of the authors' achievements in this volume, however, is the clarity with which they have brought out all the statistical problems with which the subject bristles. The main stumbling-block to progress is, of course, the fact that so few binaries are at accurately-known distances. In default of this information for individual systems the astronomer has to fall back upon average values for selected groups of stars. It is in the selection of these groups that the mathematical fun begins. The statistically-minded are here regaled with a full discussion (though not so full that the last word can be considered said) of the effects of observational selection, of systematic and of accidental errors. The more physically-inclined are presented with the challenge provided by the close correlation which exists between stellar mass and luminosity. For those whose interest lies in computation the authors describe their own new method of treating the data provided by the observer. The last section of this admirable monograph is devoted to a comprehensive catalogue of the dynamical parallaxes of 2529 stars which should provide a rich store of homogeneous material for astronomer and mathematician alike.

A. H.

Odd Numbers. By H. McKay. Pp. 215. 7s. 6d. 1940. (Cambridge)

This book is probably intended for the " imaginary general reader " rather than the " real mathematician ". Assuming these terms may be undefined, one feels it should arouse great interest in members of each group. It is not a textbook. The author's intention is to show the interest of Arithmetic and to prove it is not the " drudge whose duty it is to do everything that is dull ". No one could think of the word " dull " in connection with this book, although the " general reader " may be tempted to skip at times on account of the frequent masses of figures which might otherwise tend to overwhelm him.

The volume starts with a section designed to illustrate the magnitudes of large numbers, and leads via a chapter on indices to a neat, if brief, introduction to logarithms. Then follow chapters on proportion and comparisons, with notes on such varied ideas as the weight of a body on different planets, the space occupied by the population of the British Isles, the mass of a snowstorm, the need for care in attributing properties to models analogous to those of their prototypes, star parallax, and the formation of right-angled triangles. Two chapters containing matters which perpetually call for emphasis are those on " The Delusive Average " and " Approximations ". Indeed it is likely that teachers daily say the things contained in them but this book should reach a larger audience.

All this shows the wide range covered, and also that there is little intrinsically new in the book—the novelty and attractiveness lies in the ingenious way the large amount of matter has been collected, and in the efficient presentation of it.

Next in the text follows a plea for greater use of mathematical tables in schools on the ground that it is better for children to construct and use these than do long-winded calculations. (The example quoted, incidentally, is of a type one imagined, and certainly hoped, our schools had left behind.)

Chapters on " Oddities of Numbers " and " Construction and Solution of Problems " are very good as far as they go. But the main section not already mentioned cannot be accorded so much praise as the rest. It is an argument

to favour the f.p.s. system of units instead of the metric. It is not with the merits of either that this review is concerned—criticism is directed at the method of argument. Two quotations must suffice : " The shopgirl stretches a length of material across the convenient measured yard, without being worried by any textbook considerations, or by the extra three and a bit inches inflicted on metrical shopgirls under the pretence of uniting scientific and commercial needs " (p. 105). " A thousand kilogrammes happens to be about 2205 pounds, so that it is very little different from the ton—another piece of luck for the metric system " (p. 108). In this section also a flaw of fact appears, viz., " The franc was to be 5 grams of gold ". The original franc may certainly have been a gold coin but the standard which took its place with the metric system was 5 gm. of silver. It is a pity to have to criticise this lapse from the standards of an otherwise excellent book.

To check all the figures would amount to a rewriting of the book ; but concentration on certain sections indicates that much care has been taken with its compilation.

The sub-title given by the author is " Arithmetic Revisited ". A copy should certainly be in every school library ; and others as well as members of the Mathematics side will enjoy a visit. F. W. K.

A Course of Analysis. By E. G. Phillips. Second edition. Pp. viii, 361. 16s. 1940. (Cambridge)

The second edition of Mr. Phillips' useful textbook for first-year honours students differs from the first only in certain minor details of exposition. The author has not attempted to extend the field, since his experience has convinced him that the book " is already difficult enough for those who are using it as their first introduction to rigorous analysis ". The rapid formalism of the opening chapter, on number, breeds doubt that this may be too difficult for such students unless very considerably diluted by class-room commentary.

The middle section, on the elements of the calculus, is a little dull ; one can hardly tell the story of Moses in the bulrushes in a new way. But it is redeemed by an excellent chapter (VI) on inequalities, complete enough for most students and, for the few who will need more, an adequate introduction to Hardy, Littlewood and Pólya.

In the concluding chapters the author has grappled with multiple integrals and deserves our thanks for adding to the very few expositions of this topic available in English. He has produced a simple account of the main lines of the theory, but has also been at pains to point out the difficulties which beset any attempt to translate the appropriate geometrical concepts into terms of pure analysis, adding references to enable those who wish to do so to follow the matter further. T. A. A. B.

1330. In London, chess-players resorted to Slaughter's Coffee House (founded by John Slaughter, 1692, later often called Old Slaughter's, pulled down 1843-4) in St. Martin's Lane, and this was the headquarters of English chess from 1700-1770. Here, to a private room, came for their chess . . . and Abraham de Moivre (b. 1667, d. 1750), the mathematician, who lived for nearly thirty years on the petty sums he made at Slaughter's by chess.—H. J. R. Murray, *A History of Chess*. The authority for this anecdote is D. W. Fiske, *Notes and Queries*, 9th Ser., x, 41 (19th July, 1902). [Per Professor H. T. H. Piaggio.]

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